NONLOCAL ELASTICITY COUPLED WITH NONLOCAL DAMAGE

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ABSTRACT

In the framework of irreversible thermodynamics of nonlocal continua, the Clausius-Duhem inequality enriched by the addition of two (nonlocality) energy residuals (one for elasticity, the other for damage) is employed to devise a coupled nonlocal elastic/nonlocal damage phenomenological constitutive model. With a particular choice of the nonlocal variables (average of the strain difference and of the kinematic internal variable (k.i.v) difference), the constitutive model turns out to satisfy the criterion in virtue of which, in the case of macroscopically uniform strain and k.i.v., *both energy residuals vanish identically* hence all the state equations collapse into their local forms and as consequence all the state variables (stress, energy release force, damage hardening force) take on their respective local values. Furthermore, the inhomogeneity of the elastic moduli tensor, caused by damage, reflects on the greater attenuation effects upon the long distance particle interactions, which are accounted for through the *equivalent distance*. This replaces the Euclidean (or geodetical) distance as argument of the attenuation function and increases with the damage variation along the optimal path (while the internal length is taken constant).

Linear isotropic damage is considered, with the nonlocality introduced through the k.i.v. controlling the related hardening effects. The above thermodynamic procedure leads to the (nonnegative) intrinsic damage dissipation density having the shape of a bilinear form. The damage evolutive law is then formulated through the normality rule, such that a maximum dissipation principle can be shown to hold.

1. INTRODUCTION

Nonlocal damage is here addressed within a constitutive ambient of nonlocal elasticity in the intent to provide a coupled formulation useful for the treatment of many problems of engineering practice, e.g. fracture mechanics and composite materials. Coupling nonlocal elasticity with (either local or nonlocal) damage is not a trivial task, but raises a few theoretical difficulties.

A first difficulty arises from the fact that the usual nonlocal elasticity theory (e.g. Eringen [1], Bažant and Jirásek [2], Polizzotto [3]) holds for (macroscopically) homogeneous materials and it incurs into thermodynamic inconsistencies if applied within a context in which the material cannot be considered homogeneous due to the degradation effects produced by damage. A second difficulty is that the Euclidean (or geodetical) distance turns out to be, in the presence of damage, inadequate to account correctly for the attenuation effects upon the long distance particle interactions, since in fact, as pointed out by Polizzotto et al. [4], the (damage induced) inhomogeneity makes the attenuation effects increase beyond the values pertaining to the undamaged (homogeneous) material, till full attenuation for the failed material.

Polizzotto et al. [4] provided a nonlocal elasticity model for microscopically inhomogeneous material (but constant internal length parameter), in which both difficulties mentioned above are overcome. The therein proposed strain-difference-based nonlocal elasticity model is centered upon the concept of *equivalence distance*, capable to account for the increased attenuation effects due to the inhomogeneities. Though the model predicts a stress that coincides with the local stress in the case of uniform strain, however the local elasticity model fails to be fully recovered correspondingly, since in fact the related nonlocality energy residual is not identically vanishing in the case of uniform strain as it instead should be in order to guarantee the complete local behaviour recovering.

The purpose of the present paper is to reconsider the above nonlocal elasticity theory with reference to the case in which the material inhomogeneity is caused by damage, and to combine it with nonlocal damage mechanics. The assumption is made that the damage may manifest itself only within a subdomain of the body, as it often occurs in practice. For simplicity, the case of linear isotropic damage is considered. Also, because of the irreversible nature of the material, a thermodynamic approach is here presented, in which some guidelines previously devised by Polizzotto [3, 5] are followed.

2. THERMODYNAMIC APPROACH

The starting point is the (positive definite) Helmholtz free energy density (at constant temperature), which is taken in the form:

$$\psi = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{D}(\omega) : \boldsymbol{\varepsilon} + \frac{1}{2}\alpha\, \mathcal{A}(\Delta\boldsymbol{\varepsilon}) : \boldsymbol{D}(\omega) : \mathcal{A}(\Delta\boldsymbol{\varepsilon}) + \psi_{in}(\xi, \mathcal{A}_d(\Delta\xi)) \tag{1}$$

where α is a material constant and furthermore

ω: damage variable ($0 \leq \omega \leq 1$);

$\boldsymbol{D}(\omega) := (1-\omega)\boldsymbol{D}_0$: damaged moduli tensor;

$\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})$: local strain difference field related to the point $\boldsymbol{x}$.

$\mathcal{A}(\Delta\boldsymbol{\varepsilon}) := \int_V g(\boldsymbol{x}, \boldsymbol{x}')[\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})]\, d\boldsymbol{x}'$: nonlocal strain difference;

$\psi_{in} = \psi_{in}(\xi, \mathcal{A}_d(\Delta\xi))$: microstructure free energy potential, a functional of the kinematic internal variable (k.i.v.) ξ;

$\Delta\xi = \Delta\xi(\boldsymbol{x}, \boldsymbol{x}') = \xi(\boldsymbol{x}') - \xi(\boldsymbol{x})$: k.i.v difference field;

$\mathcal{A}_d(\Delta\xi) := \int_{Vd} g_d(\boldsymbol{x}, \boldsymbol{x}')[\xi(\boldsymbol{x}') - \xi(\boldsymbol{x})]\, d\boldsymbol{x}'$: nonlocal k.i.v difference;

$g(\boldsymbol{x}, \boldsymbol{x}') = \bar{g}(r_{eq}/\ell)$: attenuation function (with finite support) depending on the ratio between the equivalent distance $r_{eq}(\boldsymbol{x}, \boldsymbol{x}')$ (see Section 3) and the internal length ℓ;

$g_d(\boldsymbol{x}, \boldsymbol{x}') = \bar{g}(r_{eq}/\ell_d)$: attenuation function related to damage.

The Clausius-Duhem inequality (in isothermal conditions) reads

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} + P_e + P_d \geq 0 \tag{2}$$

where P_e and P_d denote the (nonlocality energy) residuals respectively related to purely elastic and purely damage deformation mechanisms.

Considering only purely elastic deformation mechanisms (hence $\dot{\omega} \equiv 0, \dot{\xi} \equiv 0, P_d \equiv 0$), one can rewrite eqn.(1) as

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} : \boldsymbol{D}(\omega) : \dot{\boldsymbol{\varepsilon}} - \alpha\, \mathcal{A}(\Delta\boldsymbol{\varepsilon}) : \boldsymbol{D}(\omega) : \mathcal{A}(\Delta\dot{\boldsymbol{\varepsilon}}) + P_e \geq 0. \tag{3}$$

After integration over the body domain V, noting that $\int_V P_e\, dV = 0$ and making use of the Green-type identity related to the self-adjoint operator $\mathcal{A}(\Delta(\cdot))$, which can be written as

$$\int_V \boldsymbol{s} : \mathcal{A}(\Delta \boldsymbol{e})\, dV = \int_V \mathcal{A}(\Delta \boldsymbol{s}) : \boldsymbol{e}\, dV \tag{4}$$

and holds for any pair of tensors $(\boldsymbol{e}, \boldsymbol{s})$, eqn. (3) gives

$$\int_V \left\{\boldsymbol{\sigma} - \boldsymbol{D}(\omega) : \boldsymbol{\varepsilon} - \alpha\mathcal{A}\big[\Delta(\boldsymbol{D} : \mathcal{A}(\Delta\boldsymbol{\varepsilon}))\big]\right\} : \dot{\boldsymbol{\varepsilon}}\, dV \geq 0. \tag{5}$$

This having to be satisfied for any field $\dot{\boldsymbol{\varepsilon}}$ in V implies the following state equation:

$$\boldsymbol{\sigma} = \boldsymbol{D}(\omega) : \boldsymbol{\varepsilon} + \alpha\mathcal{A}\left\{\Delta\big[\boldsymbol{D}(\omega) : \mathcal{A}(\Delta\boldsymbol{\varepsilon})\big]\right\} \qquad \text{in } V. \tag{6}$$

Introducing the (symmetric) two-point kernel function

$$\boldsymbol{J}(\boldsymbol{x}, \boldsymbol{x}') := \int_V \boldsymbol{D}(\omega(\boldsymbol{z}))\, g(\boldsymbol{x}, \boldsymbol{z})\, g(\boldsymbol{x}', \boldsymbol{z})\, d\boldsymbol{z} - \big[\gamma(\boldsymbol{x})\, \boldsymbol{D}(\omega(\boldsymbol{x})) + \gamma(\boldsymbol{x}')\, \boldsymbol{D}(\omega(\boldsymbol{x}'))\big]\, g(\boldsymbol{x}, \boldsymbol{x}'), \tag{7}$$

where

$$\gamma(\boldsymbol{x}) := \int_V g(\boldsymbol{x}, \boldsymbol{x}')\, d\boldsymbol{x}', \tag{8}$$

eqn (6) can be rewritten as

$$\sigma(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) + \alpha \int_V \boldsymbol{J}(\boldsymbol{x}, \boldsymbol{x}') : \big[\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})\big]\, d\boldsymbol{x}' \tag{9}$$

which is formally as proposed in [4]. The residual P_e is found to be

$$P_e = -\dot{\boldsymbol{\varepsilon}} : \mathcal{A}\big[\Delta\big(\boldsymbol{D} : \mathcal{A}(\Delta\boldsymbol{\varepsilon})\big)\big] + \mathcal{A}(\Delta\dot{\boldsymbol{\varepsilon}}) : \boldsymbol{D} : \mathcal{A}(\Delta\boldsymbol{\varepsilon}). \tag{10}$$

Then let the entire class of elastic-damage deformation mechanisms be considered and let eqns. (6)–(10) be assumed to continue to hold. Correspondently, inequality (2) takes on the form

$$\Phi := Y\dot{\omega} - \frac{\partial\psi_{in}}{\partial\xi}\dot{\xi} - \frac{\partial\psi_{in}}{\partial\mathcal{A}_d(\Delta\xi)}\mathcal{A}_d(\Delta\dot{\xi}) + P_d \geq 0 \tag{11}$$

where Y is the thermodynamic force related to $\dot{\omega}$ (energy release for unit damage), that is, by eqn (1),

$$Y := -\frac{\partial \psi}{\partial \omega} = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{D}_0 : \boldsymbol{\varepsilon} + \alpha \frac{1}{2}\mathcal{A}(\Delta\boldsymbol{\varepsilon}) : \boldsymbol{D}_0 : \mathcal{A}(\Delta\boldsymbol{\varepsilon}), \tag{12}$$

which is another state equation. By assumption, the intrinsic damage dissipation density, Φ, has to exhibit a bilinear form in terms of independent fluxes $(\dot{\omega}, \dot{\xi})$ and of related total thermodynamic forces (Y, X), that is

$$\Phi = Y\dot{\omega} - X\dot{\xi}, \tag{13}$$

where X is the (unknown) nonlocal total thermodynamic force associated to $\dot{\xi}$. On comparison of eqn (13) with eqn (11), one has

$$P_d = \frac{\partial \psi_{in}}{\partial \xi}\dot{\xi} - \frac{\partial \psi_{in}}{\partial \mathcal{A}_d(\Delta\xi)}\mathcal{A}_d(\Delta\dot{\xi}) - X\dot{\xi}. \tag{14}$$

Integration of eqn (14) over the subdomain $V_d \subset V$ where damage is allowed to occur and applying the Green-type identity relative to $\mathcal{A}_d(\Delta(\cdot))$, one obtains:

$$\int_{V_d} \left\{ \frac{\partial \psi_{in}}{\partial \xi} + \mathcal{A}_d\left(\Delta \frac{\partial \psi_{in}}{\partial \mathcal{A}_d(\Delta\xi)}\right) - X \right\} \dot{\xi}\, dV = \int_{V_d} P_d\, dV = 0. \tag{15}$$

This, having to be satisfied for any elastic-damage mechanism and for any possible evolutive laws of damage, hence for any choice of the $\dot{\xi}$ field in V_d, gives the further state equation:

$$X = \frac{\partial \psi_{in}}{\partial \xi} + \mathcal{A}_d\left(\Delta \frac{\partial \psi_{in}}{\partial \mathcal{A}_d(\Delta\xi)}\right) \qquad \text{in } V_d \tag{16}$$

Substituting eqn (16) into eqn (14) gives the expression of P_d, namely:

$$P_d = \frac{\partial \psi_{in}}{\partial \mathcal{A}_d(\Delta\xi)}\mathcal{A}_d(\Delta\dot{\xi}) - \mathcal{A}_d\left(\Delta \frac{\partial \psi_{in}}{\partial \mathcal{A}_d(\Delta\xi)}\right)\dot{\xi} \qquad \text{in } V_d. \tag{17}$$

Observing eqns (10)and (17), it is evident that the residuals P_e and P_d vanish identically in the case of uniform ε and ξ fields; also, eqns (6), (12) and (16) show that, in this circumstance, give $\sigma = \boldsymbol{D} : \boldsymbol{\varepsilon}$, $Y = (1/2)\boldsymbol{\varepsilon} : \boldsymbol{D} : \boldsymbol{\varepsilon}$, $X = \partial\psi_{in}/\partial\xi$, that is, the local material behaviour is fully recovered. A similar result was achieved by Borino *et al.* [6] for a local elastic/nonlocal damage material model.

3. EQUIVALENT DISTANCE

As pointed out in [4] in the case of inhomogeneous material, there is an increase of the long distance attenuation effects and the Euclidean (or geodetical) distance is to be substituted by the greater *equivalent distance*. This can be achieved by multiplying, the length $p(\boldsymbol{x}, \boldsymbol{x}')$ of the generic path $\Pi(\boldsymbol{x}, \boldsymbol{x}')$ joining $\boldsymbol{x}$ with $\boldsymbol{x}'$, by an attenuation factor $\theta(\boldsymbol{x}, \boldsymbol{x}') \geq 1$, and writing the augmented distance as

$$p_a(\boldsymbol{x}, \boldsymbol{x}') := \theta(\boldsymbol{x}, \boldsymbol{x}')\, p(\boldsymbol{x}, \boldsymbol{x}'). \tag{17}$$

$\theta(\boldsymbol{x}, \boldsymbol{x}')$ is a functional of the inhomogeneities located along $\Pi(\boldsymbol{x}, \boldsymbol{x}')$. The latter can be imagined subdivided into, say, $m \geq 1$ segments in each of which the damage ω varies monotonically from ω_{k-1} (at the end $\boldsymbol{x}_{k-1}$) to ω_k (at the other end $\boldsymbol{x}_k$), ($k = 1, 2, \ldots, m$; $\boldsymbol{x}_0 = \boldsymbol{x}'$, $\boldsymbol{x}_m = \boldsymbol{x}$). Here θ is proposed with the expression

$$\theta(\boldsymbol{x}, \boldsymbol{x}') := 1 + \sum_{k=1}^{m} C_k \frac{|\eta_{k-1} - \eta_k|}{\sqrt{\eta_{k-1}\eta_k}} \tag{19}$$

where $\eta := 1 - \omega$ denotes the *integrity* of the material and the C_k are nondimenisional constants. The equivalent distance $r_{eq}(\boldsymbol{x}, \boldsymbol{x}')$ is given by

$$r_{eq}(\boldsymbol{x}, \boldsymbol{x}') = \min_{\{\Pi(x,x')\}} \left\{ \theta(\boldsymbol{x}, \boldsymbol{x}')\, p(\boldsymbol{x}, \boldsymbol{x}') \right\} \tag{20}$$

which is not smaller than the geodetical distance given by

$$r(\boldsymbol{x}, \boldsymbol{x}') = \min_{\{\Pi(x,x')\}} p(\boldsymbol{x}, \boldsymbol{x}'). \tag{21}$$

Note that if $\eta(\boldsymbol{x}) = 0$, hence $\omega(\boldsymbol{x}) = 1$ and $\boldsymbol{D}(\omega)(\boldsymbol{x}) = \boldsymbol{0}$ at any point $\boldsymbol{x} \in V_d$, there it is $\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{0}$. In fact, any path $\Pi(\boldsymbol{x}, \boldsymbol{x}')$ joining $\boldsymbol{x}'$ with the considered $\boldsymbol{x}$ is characterized by $\theta(\boldsymbol{x}, \boldsymbol{x}') = \infty$, such that $r_{eq}(\boldsymbol{x}, \boldsymbol{x}') = \infty$ hence $g(\boldsymbol{x}, \boldsymbol{x}') = \bar{g}(r_{eq}/\ell) = 0 \ \forall \boldsymbol{x}' \in V$. It follows that $\boldsymbol{J}(\boldsymbol{x}, \boldsymbol{x}') \equiv \boldsymbol{0}$ by eqn (7), hence $\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{0}$ by eqn (9).

The definitions given above are formally similar to those given in [4], where η is defined as an adimensionalized scalar measure of the moduli tensor, namely $\eta = \|\boldsymbol{D}\|/d = (1 - \omega)\|\boldsymbol{D}_0\|/d$. Hence, on taking $d = \|\boldsymbol{D}_0\|$, one obtains $\eta = 1 - \omega$, as previously assumed.

4. DAMAGE EVOLUTIVE LAW

The damage evolutive law has to be formulated in terms of fluxes $(\dot{\omega}, \dot{\xi})$ and the related thermodynamic forces (Y, X) as suggested by the bilinear form, eqn (13), expressing the dissipation function Φ. Therefore, on introducing the damage condition:

$$F = F(Y, x) := Y - X - Y_0 \leq 0, \qquad \text{in } V_d \tag{22}$$

and applying the usual normality rule, one can write

$$\dot{\omega} = \dot{\xi} = \dot{\lambda} \geq 0, \qquad \dot{\lambda} F = \dot{\lambda} \dot{F} = 0 \qquad \text{in } V_d. \tag{23}$$

Here, Y is related to the strain state by eqn (12), but is independent of ω because of the particular form chosen for the Helmholtz free energy, eqn (1). By this circumstance, it may be convenient to consider the damage itself as the source of nonlocality, as pointed out by Borino *et al.* [6], but this point is not pursued here.

It can be easily shown that the above evolutive law admits a *local type* maximum dissipation principle in the form:

$$\Phi = \max_{(Y,X)} \left(Y\dot{\omega} - X\dot{\xi} \right) \qquad \text{s.t. } F(Y, X) \leq 0, \tag{24}$$

where the scalar pair $(\dot{\omega}, \dot{\xi})$ is fixed. Problem (24) provides the material state variables Y, X under which the material can undergo the damage and the damage hardening mechanisms $(\dot{\omega}, \dot{\xi})$. The nonlocal nature of the damage constitutive law emerges as soon as the field $\xi(\boldsymbol{x})$ is to be evaluated from the field $X(\boldsymbol{x})$ obtained as solution of eqn (24) at all points $\boldsymbol{x} \in V_d$. This amounts to solve the integral equation (16) with $X = X(\boldsymbol{x})$ being known.

5. COMMENTS AND CONCLUSIONS

The results of this paper can be summarized as follows:

I. The phenomenological constitutive model of coupled nonlocal elasticity/nonlocal damage herein presented exhibits the requisite to obey the local constitutive laws in the case of uniform strain and kinematic internal variable fields, which implies that, correspondingly, the energy residual vanish identically, hence the local elastic-damage behaviour is fully recovered.

II. The nonlocal-type variables used to enforce the above requisite are of the form $\mathcal{A}\big(\Delta(\cdot)\big)$, e.g. $\mathcal{A}\big(\Delta\boldsymbol{\varepsilon}\big)$ in the case of strain, that is the average of the strain difference field instead of simply the strain field. Indeed, $\Delta(\cdot)$ turns out to vanish identically whenever the $(\cdot)$ field is uniform, hence $\Delta(\cdot)$ is the nonlocal counterpart of the spatial gradient of $(\cdot)$ appearing in the gradient theories.

III. The inhomogeneity of the elastic moduli tensor produced by damage is accounted for by the *equivalence distance* which replaces the (smaller) Euclidean (or geodetical) distance $r(\boldsymbol{x}, \boldsymbol{x}')$ as argument of the attenuation function and simulates the greater attenuation effects as a consequence of damage ($r_{eq} \to \infty$ if the material fails along the path from $\boldsymbol{x}$ to $\boldsymbol{x}'$).

The present theory deserves being further studied in many aspects, in particular for a comparison of the stress-strain law here derived with analogous ones of previous formulations. Due to the lack of space, this task will be addressed elsewhere.

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