# EFFECTS OF MATERIAL GRADATION ON THE DYNAMIC STRESS INTENSITY FACTORS

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ABSTRACT

In this paper, transient elastodynamic crack analysis in two-dimensional (2-D) functionally graded materials (FGMs) is presented. An exponential material gradation is assumed for the shear modulus and the mass density. A hypersingular time-domain traction boundary integral equation method (BIEM) is applied. The method uses a convolution quadrature formula for approximating temporal convolution and a spatial Galerkin-method for spatial discretization. Numerical results are presented and discussed to analyze the effects of the material gradation on the transient elastodynamic stress intensity factors.

## 1 INTRODUCTION

Functionally Graded Materials (FGMs) have many innovative applications in engineering sciences due to their superior thermal and mechanical properties as well as improved wear- and corrosion resistances. In ideal cases, FGMs have a gradual and continuous profile of material properties instead of abrupt interfaces with discontinuous material properties like in the conventional engineering composites or laminates. One important research subject on FGMs is concerned with the transient dynamic crack analysis in such materials, which has direct applications in material sciences, fracture and damage mechanics, non-destructive material testing by ultrasonics and acoustic emission, and on-line monitoring of in-service functionally graded components or structures. Mathematically speaking, the initial-boundary value problem in transient dynamic crack analysis of FGMs is governed by a system of partial differential equations with variable coefficients, whose solution requires advanced numerical methods such as the finite element method (FEM) and the boundary element method (BEM) or the boundary integral equation method (BIEM). Though the BEM or BIEM has been applied successfully to dynamic crack analysis in homogeneous and linear elastic solids since many years, its application to dynamic crack analysis in continuously nonhomogeneous FGMs is yet still very limited. The main reason is that the dynamic Green's functions for general FGMs are either not available or too complex. This paper presents a transient dynamic crack analysis in two-dimensional (2-D) FGMs subjected to an impact crack-face loading by using a time-domain BIEM, which is an extension of the author's previous work for anti-plane dynamic crack analysis [1]. To simplify the analysis, an infinite FGM containing a finite crack is considered. An exponential material gradation is assumed for Young's modulus and the mass density of the FGMs. A numerical solution procedure is developed for solving hypersingular time-domain traction boundary integral equations (BIEs), which uses the convolution quadrature formula of Lubich [2] for approximating the temporal convolution integral, and a Galerkin-method for the spatial discretization of the BIEs. The method requires the Laplace-domain instead of the time-domain Green's functions. Numerical examples are presented and discussed to show the effects of the material gradation on the dynamic stress intensity factors and their overshoot over their corresponding static values.

## 2 PROBLEM STATEMENT AND TIME-DOMAIN BOUNDARY INTEGRAL EQUATIONS

Let us consider an infinite, isotropic, linearly elastic and continuously nonhomogeneous FGM with a finite crack of length $2a$ as shown in Fig. 1. Without body forces, the cracked FGM satisfies the following equations of motion

$$\sigma_{\alpha\beta,\beta} = \rho(\mathbf{x})\ddot{u}_\alpha \,, \tag{1}$$

the Hooke's law

$$\sigma_{\alpha\beta} = E_{\alpha\beta\delta\gamma}(\mathbf{x})u_{\delta,\gamma} \,, \tag{2}$$

the initial conditions

$$u_\alpha(\mathbf{x},t) = \dot{u}_\alpha(\mathbf{x},t) = 0 \,, \qquad t = 0 \,, \tag{3}$$

and the stress boundary conditions on the crack-faces

$$\sigma_{\alpha 2}(\mathbf{x},t) = \sigma^*_{\alpha 2}(\mathbf{x})H(t) \,, \qquad |x_1| \le a \,. \tag{4}$$

In eqns (1)-(4), $u_\alpha$ and $\sigma_{\alpha\beta}$ denote the displacement and the stress components, $E_{\alpha\beta\delta\gamma}(\mathbf{x})$ is the elasticity tensor, $\rho(\mathbf{x})$ is the mass density, $\sigma^*_{\alpha 2}(\mathbf{x})$ is the amplitude of the stress loading, and $H(t)$ is the Heaviside step function, respectively. Throughout the analysis, a comma after a quantity stands for partial derivatives with respect to spatial variables, superscript dots represent temporal derivatives with respect to time, the conventional summation rule over double indices is implied, and Greek indices take the values 1 and 2.

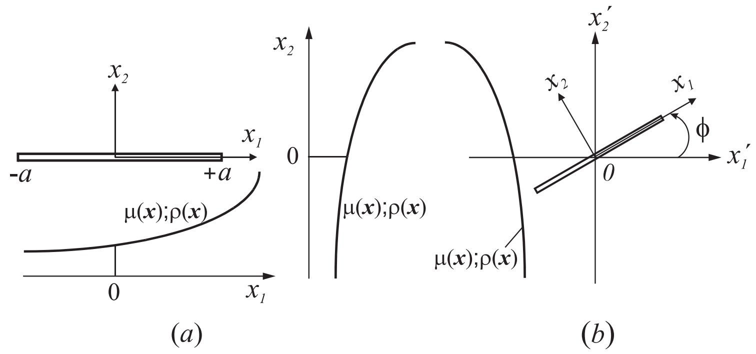


*Fig. 1: An infinite FGM containing a finite crack*

The Young's modulus $E(\mathbf{x})$ is assumed to be dependent on spatial coordinates, while the Poisson's ratio $\nu$ is taken as constant. In this case, the elasticity tensor can be expressed as

$$E_{\alpha\beta\delta\gamma}(\mathbf{x}) = \mu(\mathbf{x})E^0_{\alpha\beta\delta\gamma} \,, \quad E^0_{\alpha\beta\delta\gamma} = \frac{3-\kappa}{\kappa-1}\delta_{\alpha\beta}\delta_{\delta\gamma} + \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta} \,, \tag{5}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol. The shear modulus $\mu(\mathbf{x})$ and the constant $\kappa$ are defined by

$$\mu(\mathbf{x}) = \frac{E(\mathbf{x})}{2(1+\nu)} \,, \qquad \kappa = \begin{cases} 3-4\nu, & \text{plane strain}\,, \\ \dfrac{3-\nu}{1+\nu}\,, & \text{plane stress}\,. \end{cases} \tag{6}$$

To simplify the analysis, it is assumed that the shear modulus and the mass density have the following exponential spatial variations

$$\mu(\mathbf{x}) = \mu_0 e^{\alpha x_1 + \beta x_2}\,, \qquad \rho(\mathbf{x}) = \rho_0 e^{\alpha x_1 + \beta x_2}\,, \tag{7}$$

where $\alpha$ and $\beta$ are gradient parameters of the FGMs.
The initial-boundary value problem governed by eqns (1)-(4) can be formulated as the solution of the following hypersingular time-domain traction BIEs

$$\int_{-a}^{+a} \left[ T^G_{\gamma\alpha 2}(x_1, y_1; t)/\mu(x_1, 0) \right] * \Delta u_\gamma(y_1, t) dy_1 = \sigma^*_{\alpha 2}(x_1, 0) H(t)/\mu(x_1, 0)\,, \qquad x_1 \in [-a, +a]\,, \tag{8}$$

where $T^G_{\alpha\beta\gamma}(\mathbf{x}, \mathbf{y}; t)$ are the time-domain traction Green's functions, an $*$ stands for Riemann convolution

$$g(\mathbf{x}, t) * h(\mathbf{x}, t) = \int_0^t g(\mathbf{x}, t - \tau) h(\mathbf{x}, \tau) d\tau\,, \tag{9}$$

and $\Delta u_\alpha(x_1, t)$ are the crack-opening-displacements (CODs) defined by

$$\Delta u_\alpha(x_1, t) = u_\alpha(x_1, 0^+; t) - u_\alpha(x_1, 0^-; t)\,, \qquad |x_1| < a\,. \tag{10}$$

Note here that the required time-domain Green's functions in eqn (8) are yet not available for the considered FGMs. To circumvent this difficulty, a numerical solution procedure based on a convolution quadrature formula of Lubich [2] is developed in this analysis, which requires only the Laplace-domain instead of the time-domain Green's functions. The Laplace-domain traction Green's functions $\bar{T}^G_{\alpha 2\gamma}(\mathbf{x}, \mathbf{y}; p)$ for the considered FGMs can be expressed as the following Fourier-integral [3]

$$\bar{T}^G_{\gamma\alpha\beta}(\mathbf{x}, \mathbf{y}; p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \hat{T}^{\pm}_{\gamma\alpha\beta; j}(\xi) e^{i\xi(x_1 - y_1) \mp \gamma^{\pm}_j (x_2 - y_2)} d\xi\,, \tag{11}$$

where $\hat{T}^{\pm}_{\gamma\alpha\beta; j}(\xi) = \hat{T}^{\pm}_{\gamma\beta\alpha; j}(\xi)$, whose explicit expressions can be found in [3].

## 3 NUMERICAL SOLUTION OF THE BOUNDARY INTEGRAL EQUATIONS

To solve the hypersingular time-domain traction BIEs (8), the convolution quadrature formula of Lubich [2] in conjunction with a Galerkin-method is applied in this paper. The unknown CODs $\Delta u_\gamma(y_1, \tau)$ are approximated by the following Galerkin-ansatz

$$\Delta u_\gamma(y_1, \tau) = \sqrt{a^2 - y_1^2} \sum_{k=1}^{K} c_{\gamma;k}(\tau) U_{k-1}\left(\frac{y_1}{a}\right)\,, \tag{12}$$

where $K$ is the total number of used terms, $c_{\gamma;k}(\tau)$ are unknown time-dependent expansion coefficients and $U_{k-1}(y_1/a)$ are Chebyshev polynomials of second kind. Substituting eqn (12) into eqn (8), multiplying both sides by $\sqrt{a^2 - x_1^2} U_{l-1}(x_1/a)$, integrating them with respect to $x_1$ from $-a$ to $+a$, and applying the convolution quadrature formula of Lubich [2]

$$f(t) = g(t) * h(t) = \int_0^t g(t - \tau) h(\tau) d\tau \quad \Longrightarrow \quad f(n\Delta t) = \sum_{j=0}^{n} \omega_{n-j}(\Delta t) h(j\Delta t)\,, \tag{13}$$

a system of linear algebraic equations for the expansion coefficients is obtained as

$$\sum_{0}^{n}\sum_{k=1}^{K} A_{\gamma\alpha;kl}^{n-j} c_{\gamma;k}^{j} = f_{\alpha;l}^{n}\,, \qquad (n=1,2,...,N; \quad l=1,2,...,K)\,, \tag{14}$$

where the time-variable $t$ is divided into $N$ equal time-steps $\Delta t$, and the superscript indices stand for the time-steps. The system matrix in eqn (14) corresponds to the integration weights $\omega_{n-j}(\Delta t)$ of the convolution quadrature formula (13). The system matrix $A_{\gamma\alpha;kl}^{n-j}$ and the right-hand side $f_{\alpha;l}^{n}$ of eqn (14) are given by

$$A_{\gamma\alpha;kl}^{n-j} = \frac{r^{-(n-j)}}{M}\sum_{m=0}^{M-1}\bar{A}_{\gamma\alpha;kl}(p_m)e^{-2\pi i(n-j)m/M}\,, \tag{15}$$

$$f_{\alpha;l}^{n} = (-1)^{l+1}\int_{-a}^{+a}\left[\sigma_{\alpha 2}(x_1,0;n\Delta t)/\mu(x_1,0)\right]\sqrt{a^2-x_1^2}\,U_{l-1}\left(\frac{x_1}{a}\right)dx_1\,, \tag{16}$$

where

$$p_m = \frac{\delta(\zeta_m)}{\Delta t}\,, \qquad \delta(\zeta_m) = \sum_{j=1}^{2}\frac{(1-\zeta_m)^j}{j}\,, \qquad \zeta_m = re^{2\pi i\cdot m/M}\,. \tag{17}$$

In this paper, $M$=$N$ and $r^N$=$\sqrt{\varepsilon}$ are chosen with $\varepsilon$ being the numerical error in the computation of the Laplace-domain system matrix $\bar{A}_{\gamma\alpha;kl}(p_m)$. After some mathematical manipulations the Laplace-domain system matrix $\bar{A}_{\gamma\alpha;kl}(p_m)$ and the right-hand side can be obtained as

$$\bar{A}_{\gamma\alpha;kl}(p_m) = (\pi a)^2(kl)i^{3(k+l)}\left\{\int_0^{\infty}\left[\frac{G_{\gamma\alpha}(\xi,p_m)}{\xi^2}-\frac{G_{\gamma\alpha}^{\infty}}{\xi}\right]J_k(\xi a)J_l(\xi a)d\xi + G_{\gamma\alpha}^{\infty}\frac{\delta_{kl}}{k+l}\right\}, \tag{18}$$

$$f_{\alpha;l}^{n} = -\sigma_{\alpha 2}^{*}(\pi a)l\frac{1}{\alpha}I_l(\alpha a)\,, \tag{19}$$

where $J_k(\cdot)$ is the Bessel function of first kind and $k$-th order, $I_l(\cdot)$ is the modified Bessel function of first kind and $l$-th order, $G_{\gamma\alpha}(\xi,p_m)$ and $G_{\gamma\alpha}^{\infty}$ can be found in [3], respectively. Note here that a uniform stress crack-face loading is assumed in deriving eqn (19).

The infinite integral of (18) can be computed numerically by using an adaptive Romberg quadrature method in conjunction with the truncation method. Unlike the conventional time-domain BEM, the present time-domain BIEM applies Laplace-domain instead of time-domain Green's functions. Hence, no explicit expressions of the time-domain Green's functions are needed in the method. The evaluation of eqn (15) can be performed very efficiently by using the Fast Fourier Transform (FFT). The expansion coefficients $c_{\gamma;k}^{n}$ can be computed numerically from eqn (14) time-step by time-step. Once the time-dependent expansion coefficients $c_{\gamma;k}(t)$ have been determined numerically, the elastodynamic stress intensity factors can be computed by using

$$\left\{\begin{array}{c} K_I^{\pm}(t) \\ K_{II}^{\pm}(t)\end{array}\right\} = \frac{2}{\kappa+1}\sqrt{\pi a}\,\mu(\pm a,0)\left\{\begin{array}{c}\sum_{k=1}^{K}(\pm 1)^{k-1}kc_{2;k}(t) \\ \sum_{k=1}^{K}(\pm 1)^{k-1}kc_{1;k}(t)\end{array}\right\}, \tag{20}$$

where "$\pm$" indicates the crack-tips at $x_1$=$+a$ and $x_1$=$-a$.

## 4 NUMERICAL RESULTS AND DISCUSSIONS

The efficiency and the accuracy of the present time-domain BIEM have been verified by several numerical examples, which shows that it is sufficient to obtain very accurate results for the dynamic stress intensity factors by using the following parameters: $K$=10, $\varepsilon=10^{-12}$, and $c_T\Delta t/a$=0.1, where $c_T=\sqrt{\mu_0/\rho_0}$.

As the first example, let us consider an infinite unidirectional FGM with a crack parallel to the material gradient. The crack is subjected to an impact tensile crack-face loading of the amplitude $\sigma$. Numerical calculations have been carried out for plane strain and a Poisson's ratio $\nu$=0.3. For convenience, the following normalized dynamic stress intensity factors are introduced

$$\bar{K}_I^{\pm}(t) = K_I^{\pm}(t)/(\sigma\sqrt{\pi a}), \qquad \bar{K}_{II}^{\pm}(t) = K_{II}^{\pm}(t)/(\sigma\sqrt{\pi a}). \tag{21}$$

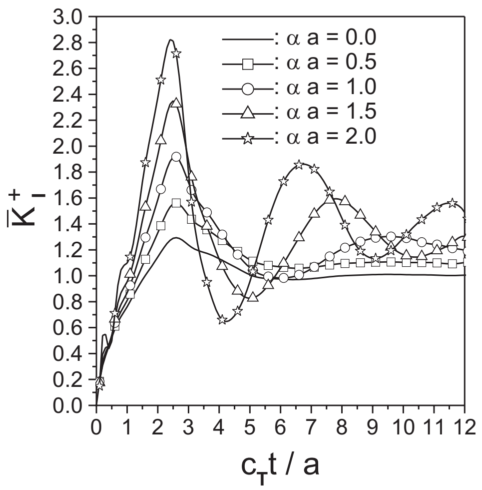


*Fig. 2: $\bar{K}_I^{+}(t)$ versus dimensionless time*

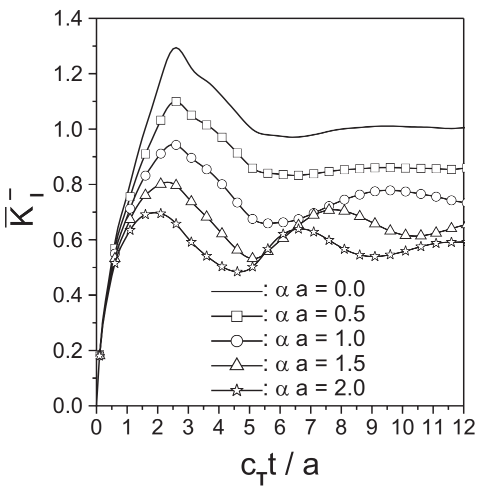


*Fig. 3: $\bar{K}_I^{-}(t)$ versus dimensionless time*

The effects of the gradient parameter $\alpha a$ on the normalized mode-I dynamic stress intensity factors are shown in Figs. 2 and 3. For this special crack orientation, the mode-II dynamic stress intensity factors are identically zero, i.e., $K_{II}^{\pm}$=0. Figures 2 and 3 show that the maximum normalized dynamic stress intensity factors at the crack-tip $x_1=+a$ increase, while those at the crack-tip $x_1=-a$ decrease with increasing gradient parameter $\alpha a$. The maximum normalized mode-I dynamic stress intensity factors at the crack-tip $x_1=+a$ are larger than those at the crack-tip $x_1=-a$, i.e., max.$\bar{K}_I^{+}$>max.$\bar{K}_I^{-}$. Compared to the homogeneous case, i.e., $\alpha a$=0, the maximum mode-I dynamic stress intensity factor at the crack-tip $x_1=+a$ is amplified, while that at the crack-tip $x_1=-a$ is reduced. Typically, the normalized mode-I dynamic stress intensity factors first increase with increasing dimensionless time, after reaching their peak values they then decrease, and thereafter they oscillate about their corresponding static values in the large-time limit.

In the next example, we consider the same crack problem as in the previous one but with an impact shear crack-face loading of the amplitude $\tau$. As in the first example, numerical calculations have been carried out for plane strain condition and a Poisson's ratio ν=0.3. The dynamic stress intensity factors are normalized by using

$$\bar{K}_I^{\pm}(t) = K_I^{\pm}(t)/(\tau\sqrt{\pi a}), \qquad \bar{K}_{II}^{\pm}(t) = K_{II}^{\pm}(t)/(\tau\sqrt{\pi a}). \tag{22}$$

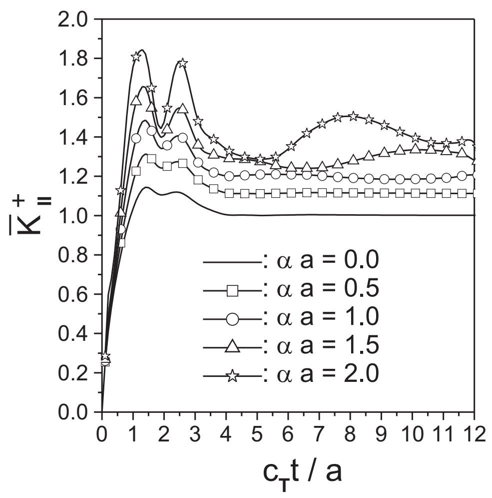


*Fig. 4: $\bar{K}_{II}^{+}(t)$ versus dimensionless time*

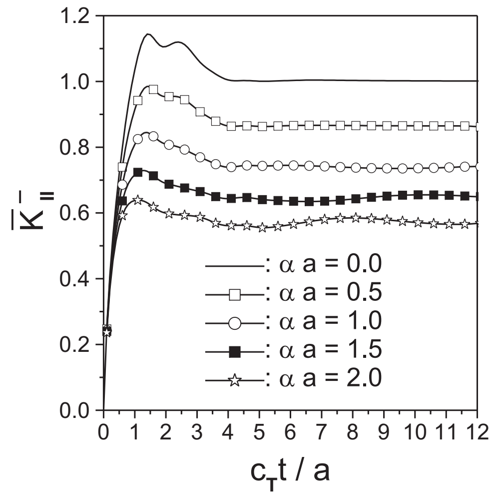


*Fig. 5: $\bar{K}_{II}^{-}(t)$ versus dimensionless time*

For five different values of the gradient parameter $\alpha a$, the normalized mode-II dynamic stress intensity factors versus the dimensionless time are presented in Figs. 4 and 4. In this case, the mode-I dynamic stress intensity factors are identically zero, i.e., $\bar{K}_I^{\pm}$=0. The maximum mode-II dynamic stress intensity factor $\bar{K}_{II}^{+}$ at the crack-tip $x_1$=$+a$ increases, while $\bar{K}_{II}^{-}$ at the crack-tip $x_1$=$-a$ decreases with increasing gradient parameter $\alpha a$. Compared to the homogeneous case, the material gradation gives rise to an increase in the $\bar{K}_{II}^{+}$-factor and a decrease in the $\bar{K}_{II}^{-}$-factor. In addition, the second peak of the normalized $\bar{K}_{II}^{+}$-factor increases with increasing gradient parameter $\alpha a$.

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