EVOLUTION OF SELF-SIMILAR CRACK SETSEVOLUTION OF SELF-SIMILAR SETS OF CRACKS. A POSSIBILITY OF LOCALISATION

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ABSTRACT
The paper considers a specific mechanism of crack growth – the crack growth under the action of pair of concentrated forced applied at the centres of the cracks. This type of loading could for instance model the action of heterogeneous stress field generated by material heterogeneities or residual strain. Cracks under such loading grow in a stable manner. If the cracks form self-similar sets then this mechanism of crack growth can either maintain or destroy self-similarity. In the first case we call the crack distribution stable, otherwise it is unstable with respect to this crack growth mechanism. It was found that if the cracks are uniformly distributed and isotropically oriented, their self-similar size distribution is stable. If, however, the cracks are all parallel to one plane, such that the cracked material becomes transversal isotropic the crack growth destroys self-similarity. It is interesting that the situation drastically changes if the parallel cracks are localised in a narrow layer because then the crack growth will maintain self-similarity. This may serve as a mechanism of localisation in the process of crack formation.

1. INTRODUCTION

In the cases when the mechanical behaviour of a material is controlled by internal microstructure encompassing a number of scales the assumption of self-similarity in distributions of microstructural elements becomes a major simplifying factor in otherwise usually intractable problem. In particular, evidence of self-similar distributions of cracks, fractures and fragments is found in such materials as concretes, rocks and the Earth's crust (e.g., Sadovskiy [1], Scholz and Aviles [2], Scholz [3], Redner [4], Olding [5], Barton and Zoback [6], Turcotte [7], Gillespie et al. [8], Yamamoto et al. [9], Dubois [10]). The appearance of self-similar structures is usually attributed to the critical state of the material (e.g., Bak and Tang [11], Chopard and Droz [12]), however the particular mechanism of formation of self-similar distributions, particularly distributions of cracks and fractures is poorly understood. The most popular approach is to consider the fractures as clusters of connected defects (e.g., Sahimi and Goddard [13], Nishiuma et al. [14], Chakrabati and Benguigui [15], Mishnaevsky [16]) which near the critical state (i.e. percolation threshold) have self-similar distributions. It should however be noted that only in the 2D picture these structures actually break the material. In real 3-D world the formation of such structures does not affect the connectedness of the body.

Dyskin [17, 18] proposed a mechanism of developing self-similar distributions of disk-like cracks based on crack interaction leading to a self-similar distribution of crack sizes with the distribution function proportional to the inverse fourth power of the crack radius. Essential in this model is the stable growth of the cracks, which is provided by a special type of loading, *viz* by a couple of concentrated forces applied at the centre of every disk-like crack. That model was only developed for isotropic crack orientations. This paper develops this approach further and considers systems of parallel cracks.

2. EMERGENCE OF SELF-SIMILAR DISTRIBUTIONS IN SETS OF INTERACTING CRACKS DRIVEN BY CONCENTRATED FORCES

Consider a material with disk-like cracks and suppose that the external static loading is such that the cracks being alone would grow in a stable manner (otherwise the first crack that starts its unstable growth will break the material). The simplest model for such a growth is loading by a pair of equivalent concentrated forces applied to the crack centre (such a loading could for instance model the action of heterogeneous stress field generated by material heterogeneities or residual strain, [17, 18]). Then the law of growth of a separate crack is given by (e.g., Tada et al, [19])

$$K_I = \frac{F}{(\pi R)^{3/2}}, \qquad K_I = K_{Ic}. \tag{1}$$

where K_I is the stress intensity factor (SIF), K_{Ic} is the fracture toughness of the material, which is assumed to be scale independent.

Suppose that the cracks are located randomly. Then, even if all cracks were initially of the same size and were loaded by exactly the same forces, the interaction will make them grow differently such that a certain size distribution of sizes will emerge. The interaction of such cracks will be modelled in the asymptotics of large distribution of sizes (Salganik [20]) assuming that: (i) cracks of close sizes do not interact directly and; (ii) the interacting cracks are very different in size. Then each crack can be considered in an equivalent medium with effective characteristics determined by all cracks of smaller sizes. As a result, for isotropic crack distribution the average SIF $\langle K_I \rangle$ for such a crack is given by (see [18] for details)

$$\frac{\langle K_I \rangle}{K_I^{\,0}} = \frac{1-\nu^2}{E}\frac{E_0}{1-\nu_0^{\,2}} \tag{2}$$

where E, ν are the effective Young's modulus, and Poison's ratio, E_0, ν_0 are the Young's modulus, and Poison's ratio of the material, K_I^0 is the SIF for the crack without interaction.

It is shown in [18] for isotropic crack distributions that as the cracks grow the difference between their sizes increases and the distribution tends to a self-similar one:

$$f(R) = \frac{27}{32MR^4}, \quad \int_{R_0}^{R_{\max}} f(R)dR = 1, \quad R_{\max}/R_0 = \left[1-(F/F_{\max})^2\right]^{-1/3}, F_{\max} = \tfrac{3}{4} K_{Ic}(2M)^{-1/2}\pi^{3/2} \tag{3}$$

where M is the number of cracks per unit volume, F_{max} is the force magnitude at which the maximum crack radius becomes infinite, which can be interpreted as the material failure. The lower boundary, R_0 corresponds to the crack growth “not assisted” by the interaction.

The emergence of self-similar distribution prompts the question whether the self-similar distributions are stable with respect to the crack growth. This question will be analysed in the following sections.

3. MECHANICS OF MATERIALS WITH SELF-SIMILAR CRACK SETS

Let the crack distribution be self-similar such that there is no characteristic size in the microstructure. According to Dyskin [21], such a material should be modelled simultaneously at many scales by a continuous set of continua (the H-continua) with the volume element sizes, H assuming all values. In this case, all continuum quantities should be also functions of scale, H. Then all characteristics of the continua become the power functions of H. Furthermore, all tensorial properties should scale isotropically, i.e. all tensorial components should scale with the same exponent [21]. In particular, the tensors of elastic moduli, **C**, and compliances, **A**, in a

Cartesian co-ordinate frame x_1, x_2, x_3 scale for the any crack orientations and any material anisotropy as

$$C_{ijkl}(H) = c_{ijkl}H^{\alpha}, \quad A_{ijkl}(H) = a_{ijkl}H^{\beta}, \quad i,j,k,l = 1,2,3, \quad \alpha = -\beta \tag{4}$$

for all non-zero components of the prefactors.

The prefactors and exponents can be determined from the following system of equations if the scaling for the contribution of cracks to the compliances $\Delta A_{ijkl} = \Delta a_{ijkl}H^{\gamma}$ is known (the dimension analysis implies that $\gamma=\beta-1$)

$$\beta a_{ijkl} = \Delta a_{ijkl} \tag{5}$$

This is generally a system of 21 equations for 22 unknowns, a_{ijkl} and β. Since the prefactors for both compliances and the increments have the same units, one of the compliance prefactors can be chosen arbitrarily, while the others and the exponent can be found from (5).

For a special case of crack distributions $f(R)=wR^{-4}$, to which distribution (3) belongs, it is shown in [21] that the assumptions of the wide distribution of sizes are satisfied and that the differential self-consistent method can be used to determine Δa_{ijkl}.

For this case, in line with (2), the average SIF scales as [21]

$$\langle K_I(H) \rangle \sim H^{-\alpha} \tag{6}$$

In the case of randomly oriented disk-like cracks the Young's modulus and Poisson's ratio scale as

$$\nu = 0, \quad E = eH^{\alpha}, \quad \alpha = -\frac{16}{9}w \tag{7}$$

4. STABILITY OF DISTRIBUTED SELF-SIMILAR SETS OF GROWING CRACKS

We are now in a position to check whether the above distribution of disk-like cracks is stable with respect to the mechanism of crack growth described in the previous section. Assuming that each homogenisation scale H~R (in the H-continuum only cracks of sizes $R>H$ can be seen) one has

$$K_{Ic} = K_I \sim R^{-\alpha-3/2}. \tag{8}$$

From here, since the fracture toughness, K_{Ic}, is assumed to be scale independent $-\alpha$-3/2=0. Then, according to (7), w=27/32 and the stable crack distribution has the form

$$f(R) = \frac{27}{32}R^{-4}, \quad V = \frac{27}{32}\ln\frac{R_{\max}}{R_{\min}} \tag{9}$$

where V is the total dimensionless concentration of cracks with sizes in the range from $R_{\min}$ to $R_{\max}$. This is a self-similar distribution coinciding with (3) given a proper relation between V and M. Thus, this type of distribution is stable with respect to the considered mechanism of crack growth.

Consider now another important case of a single set of parallel cracks with self-similar size distribution. Suppose the cracks are perpendicular to the x_3 axis. Such a set of cracks does not contribute to the components of compliances A_{1111} and A_{2222}. Therefore, Δa_{1111} =0. Then by choosing a_{1111} =1 (recall that one prefactor could be chosen arbitrarily) one has $\alpha=\beta=0$. This implies that equation (8) has no solutions. Therefore the above mechanism of crack growth will destroy the self-similarity of the distribution of parallel cracks.

5. STABILITY OF LOCALISED SETS OF PARALLEL CRACKS

The instability of self-similar distribution of parallel cracks with respect to the crack growth came form the fact that the scaling exponent vanishes. This, in its own turn, is a consequence of the fact that these cracks do not contribute to some compliances that characterise the transversal isotropic *H*-continua which model the material with one set of parallel cracks. Therefore, in order to find a stable arrangement of parallel cracks, one needs to find a situation when the cracks influence all essential components of compliances. An obvious candidate for is a localised distribution of parallel cracks, i.e. the distribution in which all cracks are concentrated within a thin layer. (In order to maintain the self-similarity the layer should be infinitesimally thin; in reality its thickness should be smaller than the lower cutoff $R_{\min}$.) Such a set of cracks can be modelled as a Winkler layer with the normal k_n and shear k_s stiffnesses defined as $\sigma_n = k_n \Delta u_n$, $\tau = k_s \Delta u_s$, where Δu_n and Δu_s are the normal and shear displacement discontinuities over the layer in response to the normal, σ_n, and shear, τ, loads.

Under the assumption of self-similar crack distribution, the stiffnesses should scale with the same exponent (since they are components of a diagonal tensor relating the stress vector and displacement discontinuity vector):

$$k_n \sim H^{\alpha}, \quad k_s \sim H^{\alpha} \tag{10}$$

Suppose the cracks are distributed as follows

$$f(R) = \frac{\omega}{R^n} \tag{11}$$

Let an *H*-continuum comprise all cracks of the size up to $R \sim H$. Transition to the scale $H+dH$ leads to adding new cracks occupying relative area $\pi d\Omega = \pi\omega R^{2-n} dR$. These new cracks increase the average stress by the factor of $(1-\pi d\Omega)^{-1}$, which results in the reduction of effective stiffness by the factor of $(1-\pi d\Omega)$. Subsequently $dk/k = -\pi\omega R^{2-n} dR$, where k stands for both normal and shear stiffnesses. Obviously, the power law is only possible if $n=3$:

$$f(R) = \omega R^{-3}, \quad \alpha = -\pi\omega, \quad k \sim R^{-\pi\omega} \tag{12}$$

From here

$$\omega = \frac{3}{2\pi}, \quad \Omega_0 = \frac{3}{2\pi} \ln \frac{R_{\max}}{R_{\min}}. \tag{13}$$

where Ω_0 is the total dimensionless concentration of cracks with sizes in the range from $R_{\min}$ to $R_{\max}$. Therefore, the self similar distribution of parallel cracks in an infinitesimally thin layer is stable with respect of crack growth under the action of concentrated forces.

6. DISCUSSION AND CONCLUSIONS

We have considered a specific mechanism of crack growth – the crack growth under the action of pair of concentrated forced applied at the centre of each crack. Such cracks grow in a stable manner. If the cracks form self-similar sets then this mechanism of crack growth can either maintain or destroy self-similarity. In the first case we call the crack distribution stable, otherwise it is unstable with respect to this crack growth mechanism. It is found that if the cracks are uniformly distributed and isotropically oriented, their self-similar size distribution is stable. If however the cracks are all parallel to one plane, such that the cracked material becomes transversal isotropic the crack growth destroys self-similarity. This implies that self-similar sets of uniformly distributed cracks cannot exist, since the self-similarity will not survive any crack growth caused, for instance, by residual stresses. It is interesting that the situation drastically changes if the parallel cracks are localised in a narrow layer. Then the crack growth will maintain self-similarity. An important question arises: could this property serve as a mechanism of localisation in the process of crack formation?

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