THE ASYMPTOTIC STESS FIELD FOR FREE EDGE JOINTS UNDER SMALL – SCALE YIELDING CONDITIONS

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ABSTRACT

In the complex engineering structures the use of bonded joints are often preferred to more traditional methods of fabrication such as bolts and welds since they are lighter and spread load more evenly. For determination the durability of such structures it is necessary to know well the stress field around stress concentrators. At the free edge of bonded joints between the adhesive and the adherend layers it is well-known that there exists an elastic singular stress field. However, little is known about the material behavior beyond the yield point. This paper presents the small-scale yielding plane-strain asymptotic field calculated for the interfacial free-edge joint singularity. The geometries are idealized as elasto - plastic materials with Ramberg - Osgood power - law hardening proprieties bonded to a rigid substrate. The solutions were obtained using a fourth order Runge - Kutta numerical method fitted to the governing equations and verified with a highly focused finite element analysis. The stress singularity order is formulated in terms of the hardening parameter and elastic solution for incompressible material.

1 INTRODUCTION

Failure of interfacial systems frequently initiates, at the free-edge joint between two materials, where a stress singularity also exists, leading to the development and propagation of an interface crack. The analysis of such interfacial free-edge stress fields is just as important, therefore, to our understanding of crack initiation and growth though in comparison to the interface crack it has received far less attention. Further, no direct link has been established between the asymptotic singular fields for the interfacial free-edge joint and the interfacial crack-tip to enable the process of crack initiation at the joint to be completely understood. Part of the reason for this is thought to be that a description of the process leading to crack initiation assuming purely elastic behaviour is complicated by the difference in stress singularity orders and fields. Indeed, it has been shown by Klingbeil & Beuth [1] that conflicting solutions are obtained if designing to prevent debond of the interfacial free-edge joint and/or to prevent propagation of an interfacial crack. Furthermore, the same limitations of the elastic solution apply to the interfacial free-edge joint, i.e. the stress and strains are unbounded. Relatively little effort has been paid to the elasto - plastic behaviour of the free-edge singularity except for the determination of plastic zone size and shape [2, 3]. There appears to have been no attempt made to understand the asymptotic elasto-plastic behaviour of the interfacial free-edge joint.

In this paper, the asymptotic structure of the elasto - plastic stress field at the interfacial free-edge joint is considered for a quarter of a Ramberg - Osgood hardening material and a rigid material bonded perfectly to form a half plane. Within the framework of small-scale yielding (SSY) the singular fields for different hardening coefficients are numerically calculated by developing asymptotic solutions to the fundamental equations of equilibrium and compatibility, and by the FE method. The asymptotic structure of the stress and displacement field developed at the bonded free-edge joint is obtained using an approach similar to that of Sharma & Aravas [4] for the interface crack. A highly focused finite element (FE) analysis provides corroborative solutions for the interfacial free-edge joint.

2 ASYMPTOTIC SOLUTION FOR THE INTERFACIAL FREE – EDGE JOINT

We consider the plane strain problem of an interface free-edge joint with the geometry presented in Figure 1. The constitutive behavior of the deformable medium is characterized by the J_2 deformation theory for a Ramberg – Osgood uniaxial stress –strain behavior:

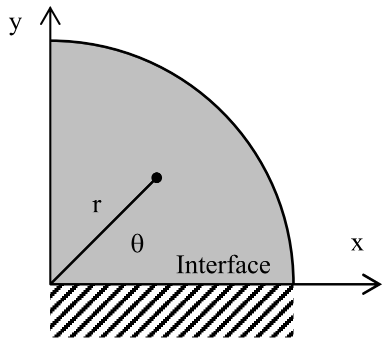


Figure 1: Interfacial free - egde joint geometry

$$\varepsilon_{ij} = \frac{1+\upsilon}{E} s_{ij} + \frac{1-2\upsilon}{3E}\sigma_{kk}\delta + \frac{3}{2}\alpha\varepsilon_o\left(\frac{\sigma_e}{\sigma_o}\right)^{n-1}\frac{s_{ij}}{\sigma_o} \quad (1)$$

where ε_{ij} is the infinitesimal strain tensor, σ_o is the yield stress, $\varepsilon_o = \sigma_o / E$ is the yield strain , the deviatoric stress is given by: $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$, the Mises equivalent stress is defined as: $\sigma_e = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2}$, n is the power-law hardening exponent $(1 \le n \le \infty)$, E is the Young's modulus, ν is the Poisson's ratio, δ_{ij} is the Kronecker delta, and α is a material constant.

To obtain the asymptotic solution the problem is formulated in terms of the leading order stresses $\tilde{\sigma}^{(0)}$:

$$\frac{\sigma(r,\theta)}{\sigma_0} = \left(\frac{\alpha\varepsilon_0\sigma_0 I_n r}{J}\right)^{s}\tilde{\sigma}^{(0)}(\theta) + Q\left(\frac{\sigma_0 r}{J}\right)^{t}\tilde{\sigma}^{(1)}(\theta) + \ldots \text{ as } \quad r \to 0 \qquad (2)$$

and the corresponding displacement leading order expansion of the form:

$$\frac{\boldsymbol{u}(r,\theta)}{\alpha\varepsilon_0} = \left(\frac{\alpha\varepsilon_0\sigma_0 I_n}{J}\right)^{sn} r^{s}\tilde{\boldsymbol{u}}^{(0)}(\theta) + \ldots \quad \text{as} \quad r \to 0 \qquad (3)$$

where $\tilde{\sigma}^{(0)}$ and $\tilde{\sigma}^{(1)}$ are normalised angular functions, $s < t < 0$, J is the J-integral, and Q is the parameter controlling the magnitude of the second term. In this expansion for the interfacial crack tip $s = - 1/(n+1)$ and the exponent t is also negative. The quantity I_n is defined [4], [5] as:

$$I_n = \int_0^{\pi}\left[\frac{n}{n+1}\tilde{\sigma}_e^{(0)n+1}\cos\theta - n_i\tilde{\sigma}_{ij}^{(0)}\left(\frac{1}{n+1}\tilde{u}_j^{(0)}\cos\theta - \frac{d\tilde{u}_j^{(0)}}{d\theta}\sin\theta\right)\right]d\theta \text{ as } \quad r \to 0 \quad (4)$$

where n_1=sinθ, n_2=cosθ and the components of stress and displacement are understood to be Cartesian rather than polar.

The eqns. (2) and (3) are substituted into the governing equations of equilibrium and stress-strain relationship (Sharma & Aravas [6, 7]). Terms having like powers of r are collected and hierarchy of problems is obtained. The leading order problem that defines $\boldsymbol{\sigma}^{(0)}$ and $\mathbf{u}^{(0)}$ consists of five non-linear ordinary differential equations:

$$\left(s+1\right)\tilde{\sigma}_{rr}^{(0)}-\tilde{\sigma}_{\theta\theta}^{(0)}+\frac{d\tilde{\sigma}_{r\theta}^{(0)}}{d\theta}=0$$
$$\frac{d\tilde{\sigma}_{\theta\theta}^{(0)}}{d\theta}+\left(s+2\right)\tilde{\sigma}_{r\theta}^{(0)}=0$$
$$\left(sn+1\right)\tilde{u}_{r}^{(0)}-\frac{3}{2}\tilde{\sigma}_{e}^{(0)n-1}\tilde{s}_{rr}^{(0)}=0 \tag{5}$$
$$\tilde{u}_{r}^{(0)}+\frac{d\tilde{u}_{\theta}^{(0)}}{d\theta}-\frac{3}{2}\tilde{\sigma}_{e}^{n-1}\tilde{s}_{\theta\theta}^{(0)}=0$$
$$\frac{1}{2}\left(\frac{d\tilde{u}_{r}^{(0)}}{d\theta}+sn\tilde{u}_{\theta}^{(0)}\right)-\frac{3}{2}\tilde{\sigma}_{e}^{(0)n-1}\tilde{\sigma}_{r\theta}^{(0)}=0$$

A second order problem may be expressed as a linear eigenvalue problem to solve for the exponent *t* and the eigen-functions for the corresponding stresses $\sigma^{(1)}$ (and displacements $\mathbf{u}^{(1)}$). A fourth-order Runge - Kutta solution to the eqns. (5) was obtained for different values of the hardening exponent *n* using the proprietary software *Mathcad* (v.2000), distributed by Adept Scientific Ltd.). An iteration scheme was used to determine the solution *s* to the non-linear eigenvalue problem and the subsequent distributions for the stresses and displacements that satisfy eqns. (5), and the boundary conditions:

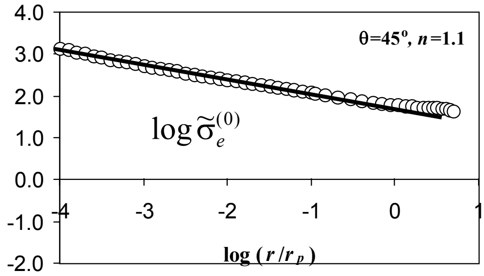
$$\sigma_{\theta\theta}\left(r,\pi/2\right)=0,\quad \sigma_{r\theta}\left(r,\pi/2\right)=0 \tag{6}$$

$$u_{r}\left(r,0\right)=0,\quad u_{\theta}\left(r,0\right)=0 \tag{7}$$

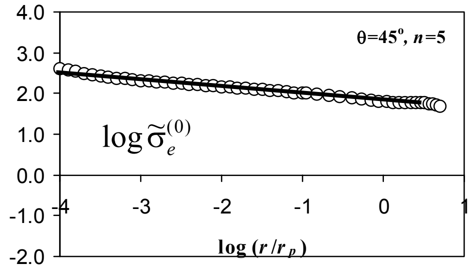
These asymptotic solutions were verified by a FE analysis performed using the software *Lusas* (v13.2, distributed by FEA Ltd., Kingston, UK). Highly - focused, refined meshes for the interfacial free-edge joint were prepared using four-noded linear elements until satisfactory convergent results were obtained. Preliminary trials to perfect the mesh were performed using the interface crack tip geometry that was compared with known solutions. The eventual FE mesh used for the interface joint problem consisted of a quarter-circle domain with boundary displacements applied calculated using the asymptotic elastic solution to the singularity problem. Logarithmic seeding was used in the radial direction consisting of 10 nodes per decibel for $-5\leq \log r/r_p \leq 1$. In the circumferential direction, the region was separated into 6 elements of uniform spacing between 0° and 9°, and 28 elements of uniform spacing between 9° and 90°. The displacements on the 0° radial were zeroed to simulate bonding to a rigid substrate and traction-free boundary conditions were assumed on the 90° radial. The authors used nine-noded Lagrangian elements and the B-bar approach to elasto - plastic analysis that seems to have been accepted in the literature as conventional.

3 RADIAL VARIATIONS IN THE ASYMPTOTIC STRESS AND STRAIN FIELDS

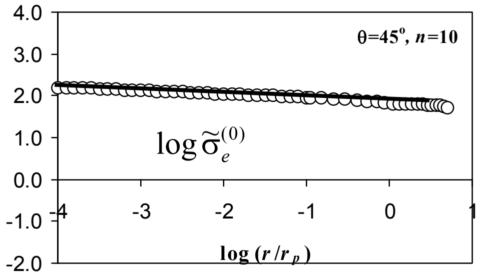
Determination of the parameter *s* by the asymptotic analysis provides the stress singularity order. To verify this hypothesis, the radial variation of the Von-Mises equivalent stress, $\tilde{\sigma}_{e}^{(0)}$, and polar component of shear strain, $\tilde{\varepsilon}_{r\theta}^{(0)}$, respectively for the four cases of hardening n = 1.1, 5, 10, and 50 are plotted in Figures 2 and 3. Nodal values of stress and strain from the FE analysis are plotted as white circles for the radial $\theta = 45^{\circ}$ in the sector $-4 \leq \log(r/r_p) \leq 1$, where r_p is the extent of the plastic zone. The asymptotic solution has been superimposed as a straight line of the appropriate gradient given by s to enable the region of singularity dominance to be determined. The singularity orders determined by the asymptotic solution are confirmed by the FE analysis since there is very good agreement between the results in general.



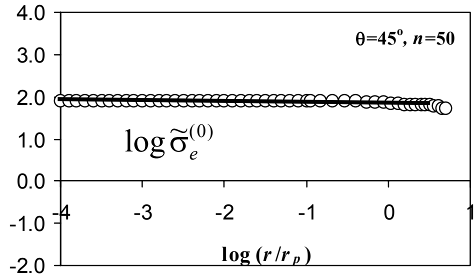
a) n=1.1



b) n=5

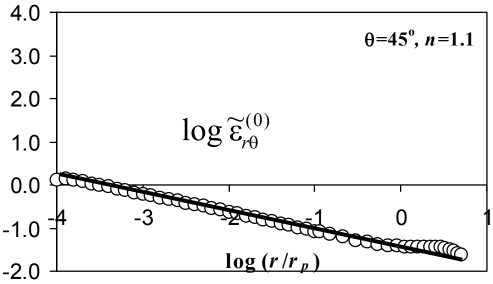


c) n=10

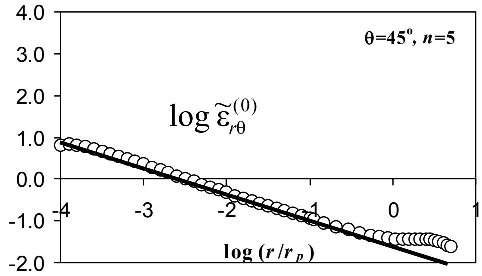


d) n=50

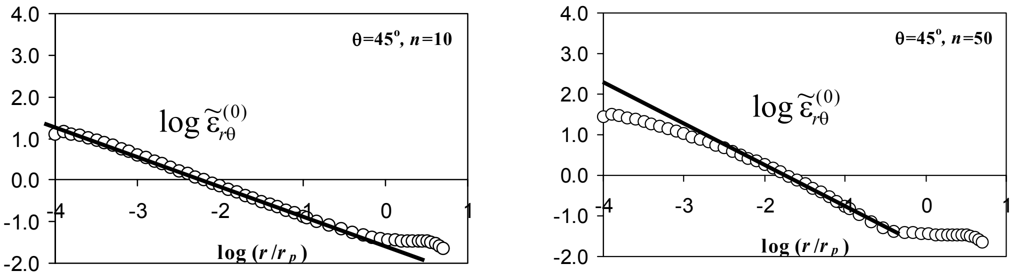
Figure 2: Radial variation of the asymptotic normalised plane-strain equivalent stresses for the interfacial free-edge at θ=45° (line=asymptotic, markers=FE).



a) n=1.1



b) n=5



c) n=10 d) n=50

Figure 3: Radial variation of the asymptotic normalised plane-strain polar shear strains for the interfacial free-edge at θ=45° (line=asymptotic, markers=FE).

The singularity orders for the interfacial free-edge joint obtained by the asymptotic analysis are shown plotted against hardening exponent *n* in the graph of Figure 4 as black squares.

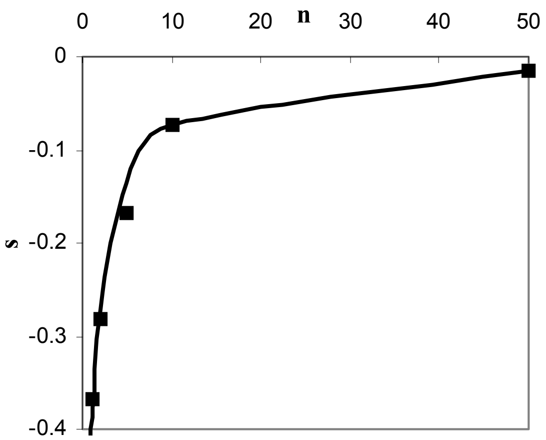


Figure 7: The singularity order *s* versus hardening exponent *n*

The elasto - plastic singularity order for the interfacial free-edge joint s_{joint} could be determined by the expression:

$$s_{\text{jo int}} = \frac{2 \cdot s_{\text{elastic}}}{1+n} \qquad (8)$$

where s_{elastic} is the elastic singularity order for the interfacial free-edge joint .
The singularity orders for the interfacial free-edge joint obtained by the asymptotic analysis appear to fit a coefficient in eqn. (8) of s_{elastic}= -0.4. The order calculated by an elastic analysis based on that of Bogy [8] gives s_{elastic}= - 0.28, and consequently would not enable eqn. (8) to give a good fit to the solutions from the asymptotic analysis. However, a remarkable fit to the asymptotic solutions using eqn. (8) is obtained if the incompressible value for Poisson's ratio is used for the upper material half, i.e. $\upsilon = 0.5$. If the singularity order is calculated by an elastic analysis based on that of Bogy [8] this yields λ - 1 = -0.4065. In other words, the singularity order for the elasto-plastic behaviour at interfacial free-edge singularity may be predicted using eqn. (8) if s_{elastic} is calculated by an elastic analysis assuming incompressible material. This is verified by plotting eqn. (8) in Figure 4 using s_{elastic} = -0.4 as shown by the solid black line. Further support for this hypothesis may be reasoned since the derivation of the governing asymptotic eqns. (5) assumes incompressible material also.

An implication of eqn. (8) is that the singularity order for the strain-energy-density is always – 0.8 for any value of *n*. This was verified by the FE analysis and the asymptotic analysis, and show good agreement

3 CONCLUSIONS

For an isotropic elasto - plastic material bonded to a rigid substrate the SSY asymptotic plane-strain behaviour at the interfacial free-edge joint has been identified for several values of the hardening exponent *n*. Using an asymptotic analysis the polar components of equivalent stress and shear strain have been determined and were confirmed by a highly - focused FE analysis. The singularity orders under elasto - plastic behaviour were identified and shown to be only dependent on the hardening exponent *n* and not on the elastic properties of the material. The singularity orders for the joint, for given *n*, could be predicted using a formula proposed by authors.

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