EFFECTIVE STRESS INTENSITY FACTORS IN MATERIALS WITH MICRODEFECTS AND MICROINCLUSIONS

NATALA B. ROMALIS
Department of Mathematics, University of Bridgeport, Bridgeport, CT 06601, USA

ABSTRACT

The present work is a continuation of the author's research on a macrocrack propagation in stochastically inhomogeneous materials (Romalis, 1975, 2003)

A stochastically inhomogeneous material is understood in a sense that elastic moduli are random functions of coordinates. The effective elastic moduli of such materials become functions of geometric parameters of solids, particularly, macrocracks. A macrocrack present in a material creates a new boundary, therefore a boundary value problem for a stochastically inhomogeneous material is considered. It was shown, using stochastic linearization, that, in a frame of linear elasticity, the average stresses have a well-known singularity at the macrocrack tip, which justified an introduction of the effective stress intensity factor (ESIF). Asymptotic solutions to the problem were obtained as power series expansion over a small parameter representing a ratio of a size of a microdefect or microinclusion to a macrocrack size, with coefficients of the series depending on statistical characteristics of microvoid or microinclusion distributions.

The effective stress intensity factors (ESIF) for randomly distributed microcracks, microvoids, and for 2-phase composite material with elastic microinclusions were obtained as functions of effective elastic moduli of material and statistical characteristics of a material structure distribution. The radius of convergence of the series solution was determined as a function of concentrations and mechanical properties of inclusions. All calculations were performed by *Mathematica*.

INTRODUCTION

The Effective Stress Intensity Factors in stochastically inhomogeneous materials are considered where stochastically inhomogeneous materials are understood in the sense that elastic moduli are random functions of coordinates. The assumption of small fluctuations of elastic moduli from their averages was made. There is a number of theoretical models which have been developed to predict the effective elastic moduli of composite materials (see, for example, review by Christensen (1991)). However, the effective elastic moduli are not the local characteristics. Since a crack forms a new boundary, a corresponding boundary value problem must be considered. The boundary value problem for the crack in stochastically inhomogeneous material for the plane problem was discussed earlier by Romalis (1975, 1999). Now we consider an estimation of the upper and lower bounds of effective stress intensity factors (SIF) and critical loads in nonhomogeneous materials.

It was shown by Romalis (1975) that in the frame of the linear elasticity means of the stresses in the vicinity of the radius r of the crack still have a well known order $(r)^{-1/2}$ of singularity which justifies the introduction of the "effective stress intensity factors (SIF)" for stochastically inhomogeneous materials with cracks. In present work, statistical characteristics of effective SIFs are determined for the plane elasticity problem of inhomogeneous material as functions of the statistical characteristics of the material.

STATEMENT OF THE PROBLEM

The closed system of the equations of the plane elasticity in heterogeneous body has a following form

$$\sigma_{ij,j} = 0;\ \Delta(\gamma\sigma_{mm}) = \sigma_{ij}\frac{\partial^2 q}{\partial x_i \partial x_j};\quad \sigma_{ij} n_j = g_i \Big|_L;$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};\ \gamma = \frac{1}{E}:\quad q = \frac{(1+\nu)}{E} \qquad (1)$$

where σ_{ij} are stresses, g_j are known deterministic functions on the boundary L, and elastic constants q and γ are introduced as functions of the Young modulus E and the Poisson ratio ν.

Let $q(x,y)$ and $\gamma(x,y)$ be random functions of coordinates. Then the problem (1) is statistically nonlinear with respect to the random functions σ_{ij}. Problem (1) can be linearized if we assume that functions $q(x,y)$ and $\gamma(x,y)$ are statistically homogeneous, that is, their means: $\langle q\rangle = const;\quad \langle\gamma\rangle = const$, and can be represented as sums of their means and perturbations. We seek a solution to the problem as a power series over the parameter λ in the following form

$$q = \langle q\rangle + \lambda q';\quad \gamma = \langle\gamma\rangle + \lambda\gamma';\quad \sigma_{ij} = \sum_{k=0}^{\infty}\lambda^k\sigma_{ij}^{(k)} \tag{2}$$

SERIES SOLUTION

Introducing complex variable z and substituting (2) into (1), and equating factors of the same powers of λ, we can obtain the boundary value problem for initial approximation

$$\langle\gamma\rangle\Delta(\sigma_x^0 + \sigma_y^0) = 0; \text{ (compatibility conditions)}$$

$$(X_n^0 + Y_n^0)_L\Big| = g_1 + ig_2; \quad \text{(boundary conditions)} \tag{3}$$

$$\frac{\partial}{\partial z}(\sigma_x^0 - \sigma_y^0 + 2i\sigma_{xy}^0) + \frac{\partial}{\partial z}(\sigma_x^0 + \sigma_y^0) = 0; \text{ (equilibrium conditions)}$$

and recurrence sequence of statistically linear boundary value problems for successive approximations of the stresses

$$\frac{\partial}{\partial z}(\sigma_x^{(k)} - \sigma_y^{(k)} + 2i\sigma_{xy}^{(k)}) + \frac{\partial}{\partial \bar z}(\sigma_x^{(k)} + \sigma_y^{(k)}) = 0; \text{ (equilibrium conditions)}$$

$$\langle\gamma\rangle\Delta(\sigma_x^{(k)} + \sigma_y^{(k)}) = f^{(k-1)}; \quad \text{(compatibility conditions)} \tag{4}$$

$(X_n^{(k)} + Y_n^{(k)})_L\Big| = 0;$ (boundary conditions), where expressions $f^{(k-1)}$ consist of the sums of products of the following type $\sigma_{ij}^{(k-1)}\partial^2 q/\partial z^2$.

The boundary value problem (3) is a problem for the homogeneous solid with a crack loaded with a given stresses, while recurrence relations (4) are the boundary value problems for a crack free of tractions in a solid with distributed body forces. Let us consider the infinite plane (in case of the plane stress), or an infinite cylindrical body (for the plane strain) with a crack of the length $2l$ whose banks are loaded with the given forces. Using the Airy function $U(x, y)$, the problem (4) can be reduced to a biharmonic inhomogeneous equation for each approximation

$$\sigma_x^{(k)} = \frac{\partial^2 U^{(k)}}{\partial y^2};\quad \sigma_y^{(k)} = \frac{\partial^2 U^{(k)}}{\partial x^2};\quad \sigma_{xy}^{(k)} = -\frac{\partial^2 U^{(k)}}{\partial x\partial y};$$

$$\Delta^2 U^{(k)} = f^{(k-1)};\quad (X_n^{(k)} + iY_n^{(k)})\Big|_L = 0 \tag{5}$$

Solution to the problem (5) was obtained as a sum of solutions to two sub-problems: (I) the inhomogeneous problem for the infinite body without the crack, and (II) homogeneous problem for the infinite body with a cut loaded with stresses opposite to those acting on the cut trace from the problem (I). The problem (I) was solved using the Green's function for the biharmonic equation for the infinite plane. The problem (II) for the infinite plane with a linear cut on [$-l$, l] can be written in the following form

$$\Delta^2 U^{(k)} = 0; \quad (X_n^{(k)} + iY_n^{(k)})\Big|_L = -(X_{n1}^{(k)} + iY_{n1}^{(k)})\Big|_L;$$

where ($X_n^{(k)}, Y_n^{(k)}$) and ($X_{n1}^{(k)}, Y_{n1}^{(k)}$) represent, correspondingly, resultant vectors of the forces applied to the cut line, obtained from the solution of the problem (I), was solved by the Muskelishvili (1963). The complex potentials for the k –th approximation were found in the following form

$$\Phi^{(k)}(z) = -\frac{1}{2\pi}\int_{-l}^{l}\frac{\sqrt{t^2 - l^2}(\sigma_{y1}^{(k)} - \sigma_{xy1}^{(k)})dt}{t - z}$$

where $(\sigma_{y1}^{(k)} - \sigma_{xy1}^{(k)})$ are stresses on the trace of the cut that was found by the Airy function as a solution of the non-homogeneous biharmonic equation. It was shown by the asymptotic analysis that means of the complex potentials preserved their ordered of singularity of $(r)^{-1/2}$ at the tips of the crack, which justifies introduction of the effective stress intensity factor (SIF) in the non-homogeneous solid with a crack whose k –th approximation has a form

$$\left\langle K^{(k)} \right\rangle = \left\langle K_I^{(k)} - iK_{II}^{(k)} \right\rangle = \frac{i}{8\pi\sqrt{l}}\int_{-l}^{l}\iint_S \sqrt{\frac{t+l}{t-l}}\left\langle f^{(k-1)}(\xi,\eta)\right\rangle\left\{\frac{(t-\xi)^2 - \eta^2}{(t-\xi)^2 + \eta^2} + \ln\left|(t-\xi)^2 + \eta^2\right|\right\} d\xi d\eta dt \quad (6)$$

The statistical characteristics of the material's elastic moduli are involved into expression $\left\langle f^{(k-1)} \right\rangle$. The further analysis was conducted under the assumption that elastic moduli's fluctuations are normally distributed. Then the standard deviations of the effective SIF's were derived in the integral form, analogous to (6). We consider two-phase material with small elastic inclusions. The compliances of the composite material were used in the following form suggested by Dundurs and Jasiuk (1996)

$$S = S_1 + c(S_2 - S_1)\frac{1+\alpha}{1+\beta}; \quad S = \frac{1}{\mu}; \quad \alpha = \frac{\Gamma(\kappa_1 + 1) - (\kappa_2 + 1)}{\Gamma(\kappa_1 + 1) + \kappa_2 + 1};$$
$$\beta = \frac{\Gamma(\kappa_1 - 1) - (\kappa_2 - 1)}{\Gamma(\kappa_1 + 1) + \kappa_2 + 1}; \quad \Gamma = \frac{\mu_2}{\mu_1}; \quad (7)$$

where indices "1" and "2" denote the different phases, and c is a random fraction of the inclusions. For voids or microcracks the following expression for elastic moduli was used $E = E_1 c + E_2(1-c)$ with mean of the value of c evaluated by $\langle c \rangle = \omega\frac{Na^2}{A} = n\frac{\alpha^2}{l^2}$ where ω is the average density of microdefects, n is an average number of microdefects in a l x l square, and N is the total number of microdefects in a body with a cross-section A. Substituting the statistical characteristics of compliances into (6) mean and the standard deviation of the ESIF's were obtained. Iw was shown that statistical moments of odd order were equal to zero. Using the correlation theory, statistical moments of even orders can be introduced through the correlation function in the assumption that ordinates of the fluctuation of compliances are normally distributed.

By the Erdogan and Sih(1963) criterion a crack starts propagating in the direction perpendicular to the direction of the maximum tensile loads was used. The angle of the initial expanding was assumed be equal to

$$\theta^* = 2\arctan\frac{K_I - \sqrt{K_I^2 + 8K_{II}^2}}{4K_{II}} \quad (8)$$

with a corresponding fracture criterion

$$\cos^3\frac{\theta^*}{2}(K_I - 3K_{Ii}\tan\frac{\theta^*}{2}) = \frac{K_{IC}}{\sqrt{\pi}} \quad (9)$$

where K_{IC} is a fracture constant. Formulas (8) and (9) can be easily combined to determine the dimensionless critical load

$$F(K_I, K_{II}) = p\sqrt{l} / K_{Ic} \quad (10)$$

of the crack extension, where p is an external load, and l is a crack length. . Assuming that both modes of the stress intensity factors are normally distributed, and considering a left side of (10) as a function of two random variables, mean and standard deviation of the function F were determined as a function of statistical characteristics of inclusions, and the inclusion/crack ratios.

Using the Tchebyshev's theorem, the upper and lower bounds of the average of the dimensionless critical load were determined as a function of geometry and statistical characteristic of inclusions.

CONCLUSION

Suggested approach allows to obtain an approximation of the bounds of mean of the effective fracture loads in stochastically inhomogeneous materials. Another approach to modeling a macrocrack in stochastically inhomogeneous material consists of study of an explicit interaction of the macrocrack with a microcrack, micropore or microinclusion (see review by Tamuzs *at al* (2000). One of such model was developed by Romalis and Tamuzs (1984), and applied to various problems in (Tamuzs et al (2000). Modeling a crack in non-homogeneous solid and assuming a desired density of microdefects generated randomly, means of the critical loads were found on the basis of n such generated realizations. Review of other approaches is given by Petrova et al (2000).

REFERENCE

Christensen, R.M. (1991): *Mechanics of Composite Materials*, Kriger Publ. Co.

Dundurs, J., and I..Jasiuk (1996): "Effective Elastic Moduli in Composite Materials: Reduced Parameter Dependence*", Applied Mechanics in the Americas,* Proceedings of PACAM V, Vol.4, pp.167-170.

Erdogan, F., and Sih, G.C. (1963): " On the Crack Extention in Plates under Plane Loading and Transverse Shear", Trans ASME, J.Basic Engineering, Vol .85(4), pp.519-528.

Muskelishvili, N.I. (1963): *Some basic Problems of the Mathematical Theory of Elasticity*, Noordhoff Ltd.

Petrova, V., Tamuzs, V., and Romalis, N. (2000): "A Survey of the Macro-Microcrack Interaction Problems", *Applied Mechanics Review*, Vol.53, No.5, pp. 117-146.

Romalis, N.B. (1975): Effective Stress Intensity Factors at the Crack Tip in the Stochastically Inhomogeneous Solid", *Applied Mechanics (Prikladnaya Mechanika),* Vol.11, No. 11, pp.100-105.

Romalis, N. (2003): "On the Crack Arrest in Stochastically Inhomogeneous Materials", in the Proceedings of ICCES03.

Romalis, N.B., and V.P. Tamuzs (1984): "Propagation of a Main Crack in a Body with Distributed Microcracks", Mechanics of Composite Materials, 20(2), pp.35-43.

Tamuzs, V., Romalis, N., and Petrova, V. (2000*): Fracture of Solids with Microdefects*, Nova Science Publishers, Inc., New York