ANISOTROPIC PLANE THAT CONTAINS A CIRCULAR INCLUSION WITH IMPERFECT INTERFACE

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ABSTRACT

The plane problem for an infinite plane that contains a circular inclusion with an imperfect interface is studied. The explicit solutions are derived based on Stroh formalism and the material constants of the matrix and the inclusion are both anisotropic and different from each other. The model for the matrix/inclusion interface is considered as a spring-type layer with vanishing thickness such that the interface effect can be described by continuous tractions but discontinuous displacements across the interface layer. By using analytic continuation in a dual coordinate system, two interfacial conditions lead to the simultaneous equations for two analytical vector functions. The two vector functions which are the main part of the expressions of displacement and stress function are expanded in forms of Laurent's series. Thereafter, the series coefficients are determined by comparing the coefficients whenever the terms associated with the external load are at hand. The methodology for this research is illustrated by solving an example that the matrix is subjected to a remote uniform stresses field and the three interfacial parameters of homogeneous imperfect interface are identical. The simple and straightforward features of the application of our results reveal the applicability to those problems which is under more complicate external loading.

1 INTRODUCTION

In the real situation, the inclusion/matrix interface in the composite materials can be considered of the concept for imperfect interface due to the existence of interface damage. Hashin [1] solved the problem for a body contains a spherical inclusion. Ru and Schiavone [2] made an extensive literature review for the problems for inclusion problems with imperfect interface. Ru [3] presented a general method for the problem of a circular inclusion with a circumferentially inhomogeneous sliding interface in plane elastostatics. Kattis and Providas [4] use a superposition method and the holomorphic transformation of a complex function regarding a circular boundary to present an elastic field for an inclusion/matrix system with spring-type imperfect interface. Sudak et. al. [5] presented a solution method for the circular inclusion problem. In this paper, the normal and tangential spring-factor type interface parameters are circumferentially inhomogeneous and set to be identical. Shen et. al. [6] considered an elliptic inclusion problem with homogeneous imperfect interface. Recently, Schiavone [7] provided the solutions of the problem associated with a circular elastic inhomogeneity embedded within an infinite matrix in antiplane shear. Nevertheless, the problems in the above literature are all isotropic ones. As this author's knowledgement, there is no paper discussing the anisotropic inclusion problem with imperfect interface found in the literature.

In this paper, we present explicit solutions for the problem associated with a generally anisotropic circular inclusion embedded within an infinite generally anisotropic plane. The materials constants of the inclusion and the matrix can be completely different from those of each other and that the inclusion/matrix interface is imperfect. In order to demonstrate the applicability of our results, an example that the matrix is subjected to remote uniform stress field is solved.

2 FORMULATION

On a coordinate system x_i, i=1,2,3, the displacement vector $\mathbf{u}=[u_1, u_2, u_3]^T$ and stress function vector $\phi=[\phi_1, \phi_2, \phi_3]^T$ for plane thermoelastic problems can be expressed as follows (Ting [8])

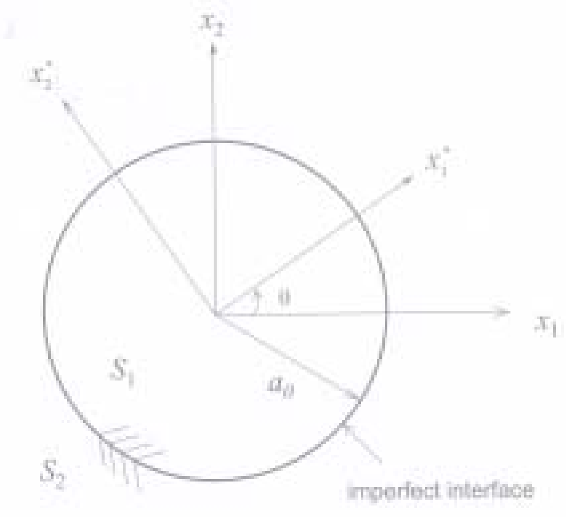

Figure 1. A circular inclusion/matrix system with imperfect interface

$$\mathbf{u}=2\mathrm{Re}\{\mathbf{A}\mathbf{f}(z_\alpha)\}\,;\phi=2\mathrm{Re}\{\mathbf{B}\mathbf{f}(z_\alpha)\}\,, \tag{1}$$

in which **A** and **B** are Stroh matrices and a function vector $\mathbf{f}(z_\alpha)$ are

$$\mathbf{f}(z_\alpha)=\begin{bmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3)\end{bmatrix}^T. \tag{2}$$

The notation $[\ ]^T$ is denoted for the transpose of a matrix. The parameters in $f_\alpha(z_\alpha)$, are $z_\alpha=x_1+p_\alpha x_2$, α=1,2,3, with p_α being the elasticity eigenvalues whose imaginary parts are positive. The three functions $f_\alpha(z_\alpha)$ are considered of the same function form.

Consider an anisotropic matrix containing a circular inclusion that the material of inclusion is different from that of the matrix (see Figure 1). The regions for the inclusion and the matrix are respectively denoted by S_1 and S_2 and all the function quantities associated with them are denoted by the subscripts (1) and (2), respectively. Assume that the inclusion be bonded to the matrix by a spring-constant-type layer that the interface conditions are given by

$$\|\sigma_{ri}(a_0,\theta)\|=0,\ \text{ and }\ \sigma_{ij}(a_0,\theta)=h_j\|u_i(a_0,\theta)\|;\, i,\, j=r,\theta,z. \tag{3}$$

where i, j stand for r, θ, and z, $\|*\|=(*)_2-(*)_1$ denotes for the function value jump across the interface and the three interfacial parameter h_r, h_θ, h_z, are nonnegative. Since

$$u_r=\mathbf{n}^T\mathbf{u}\,;u_\theta=\mathbf{m}^T\mathbf{u}\,;\ \ u_z=(\mathbf{u})_3; \tag{4}$$

$$\sigma_{rr}=\mathbf{n}^T\mathbf{t}_r\,;\sigma_{r\theta}=\mathbf{m}^T\mathbf{t}_r\,;\ \ \sigma_{rz}=(\mathbf{t}_r)_z\,, \tag{5}$$

where $\mathbf{n}^T=\begin{bmatrix}\cos\theta, & \sin\theta, & 0\end{bmatrix}$, $\mathbf{m}^T=\begin{bmatrix}-\sin\theta, & \cos\theta, & 0\end{bmatrix}$, $\mathbf{t}_r=-\partial\phi/r\partial\theta$. Eqns (4) and (5) can be represented by

$$\mathbf{u}_r=\mathbf{\Omega}\mathbf{u}\,,\ \ \boldsymbol{\sigma}_r(a_0,\theta)=\mathbf{\Omega}\mathbf{t}_r(a_0,\theta)\,, \tag{6}$$

where $\boldsymbol{\sigma}_r=[\sigma_{rr}(a_0,\theta),\ \sigma_{r\theta}(a_0,\theta),\ \sigma_{rz}(a_0,\theta)]^{\mathrm{T}}$, $\mathbf{u}_r=[u_r(a_0,\theta),\ u_\theta(a_0,\theta),\ u_z(a_0,\theta)]^{\mathrm{T}}$ and

$$\mathbf{\Omega}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{7}$$

Hence, eqn (3) can be equivalently expressed by

$$\|\phi(a_0,\theta)\|=\mathbf{0}\,;\ \ \mathbf{\Theta}\boldsymbol{\sigma}_r(a_0,\theta)=\|\mathbf{u}_r(a_0,\theta)\|, \tag{8}$$

in whch, the diagonal matrix $\mathbf{\Theta}=<<1/h_\alpha>>,\alpha=r,\theta,z.$

Assume that the stress function ϕ and the displacement **u** for regions S_1 and S_2 are

$$\mathbf{u}_{(2)}=\mathbf{u}^\infty+2\,\mathrm{Re}[\mathbf{A}_{(2)}\mathbf{f}_{(2)}(z_{(2)})];\quad \mathbf{u}_{(1)}=2\,\mathrm{Re}[\mathbf{A}_{(1)}\mathbf{f}_{(1)}(z_{(1)})]; \tag{9}$$

$$\phi_{(2)}=\phi^\infty+2\,\mathrm{Re}[\mathbf{B}_{(2)}\mathbf{f}_{(2)}(z_{(2)})];\quad \phi_{(1)}=2\,\mathrm{Re}[\mathbf{B}_{(1)}\mathbf{f}_{(1)}(z_{(1)})]. \tag{10}$$

Since the three interfacial points z_α, α=1~3, are three separate points in the x_i coordinate system, therefore, it could not be easy to determine the two function vectors $\mathbf{f}_{(j)}(z_{(j)})$, j=1,2, by the interfacial conditions. In order to conquer this difficulty, we can use the following transformations

$$\zeta_{\alpha(j)} = [z_{\alpha(j)} + \sqrt{z^2_{\alpha(j)} - a_0^2(1+p^2_{\alpha(j)})}\,]/[a_0(1-ip_{\alpha(j)})], \alpha = 1 \sim 3,\ j = 1,2. \tag{11}$$

The three point $\zeta_{\alpha(j)}$, can merge to a single genetic point on the ζ-plane, say $\zeta_1=\zeta_2=\zeta_3=\zeta$.

The two vector functions $\mathbf{f}_{(j)}(z_{(j)})$, j=1,2, can be expanded in Laurent's series and expressed by

$$\mathbf{f}_{(2)}(z_{\alpha(2)}) = \ll \sum_{m=0}^{\infty} \eta_{m\alpha}\zeta^{-m}_{\alpha(2)} \gg = \sum_{m=0}^{\infty} \ll \zeta^{-m}_{\alpha(2)} \gg \boldsymbol{\eta}_m, \tag{12}$$

$$\mathbf{f}_{(1)}(z_{\alpha(1)}) = \ll \sum_{m=-\infty}^{\infty} \lambda_{m\alpha}\zeta^{m}_{\alpha(2)} \gg = \sum_{m=-\infty}^{\infty} \ll \zeta^{m}_{\alpha(2)} \gg \boldsymbol{\lambda}_m, \tag{13}$$

where $\boldsymbol{\eta}_{\mathrm{m}}=[\eta_{\mathrm{m}1},\ \eta_{\mathrm{m}2},\ \eta_{\mathrm{m}3}]^T$ and $\boldsymbol{\lambda}_{\mathrm{m}}=[\lambda_{\mathrm{m}1},\ \lambda_{\mathrm{m}2},\ \lambda_{\mathrm{m}3}]^T$ are constant vectors to be determined. Substituting eqns (9), (10), (12) and (13) into the interfacial conditions (8) leads to

$$-\boldsymbol{\Theta\Omega}(\partial\phi^{\infty}/r\partial\theta) + \sum_{m=1}^{\infty}[(im/a_0)\boldsymbol{\Theta\Omega}\mathbf{B}_2\boldsymbol{\eta}_m\zeta^{-m}] - \sum_{m=1}^{\infty}[(im/a_0)\boldsymbol{\Theta\Omega}\overline{\mathbf{B}}_2\overline{\boldsymbol{\eta}}_m\zeta^{m}]$$

$$= \boldsymbol{\Omega}\mathbf{u}^{\infty} + \sum_{m=1}^{\infty}(\boldsymbol{\Omega}\mathbf{A}_2\boldsymbol{\eta}_m - \boldsymbol{\Omega}\overline{\mathbf{A}}_1\overline{\boldsymbol{\lambda}}_m - \boldsymbol{\Omega}\mathbf{A}_1\boldsymbol{\lambda}_{-m})\zeta^{-m} + \sum_{m=1}^{\infty}(\boldsymbol{\Omega}\overline{\mathbf{A}}_2\overline{\boldsymbol{\eta}}_m - \boldsymbol{\Omega}\mathbf{A}_1\boldsymbol{\lambda}_m - \boldsymbol{\Omega}\overline{\mathbf{A}}_1\overline{\boldsymbol{\lambda}}_{-m})\zeta^{m}$$

$$+ \boldsymbol{\Omega}\mathbf{A}_2\boldsymbol{\eta}_0 + \boldsymbol{\Omega}\overline{\mathbf{A}}_2\overline{\boldsymbol{\eta}}_0 - \boldsymbol{\Omega}\mathbf{A}_1\boldsymbol{\lambda}_0 - \boldsymbol{\Omega}\overline{\mathbf{A}}_2\overline{\boldsymbol{\lambda}}_0, \tag{14}$$

$$\boldsymbol{\Omega}\phi^{\infty} + \sum_{m=1}^{\infty}(\boldsymbol{\Omega}\mathbf{B}_2\boldsymbol{\eta}_m - \boldsymbol{\Omega}\mathbf{B}_1\boldsymbol{\lambda}_{-m} - \boldsymbol{\Omega}\overline{\mathbf{B}}_1\overline{\boldsymbol{\lambda}}_m)\zeta^{-m} + \boldsymbol{\Omega}\mathbf{B}_2\boldsymbol{\eta}_0 + \boldsymbol{\Omega}\overline{\mathbf{B}}_2\overline{\boldsymbol{\eta}}_0 - \boldsymbol{\Omega}\mathbf{B}_1\boldsymbol{\lambda}_0 - \boldsymbol{\Omega}\overline{\mathbf{B}}_1\overline{\boldsymbol{\lambda}}_0$$

$$= \sum_{m=1}^{\infty}(\boldsymbol{\Omega}\mathbf{B}_1\boldsymbol{\lambda}_m + \boldsymbol{\Omega}\overline{\mathbf{B}}_1\overline{\boldsymbol{\lambda}}_{-m} - \boldsymbol{\Omega}\overline{\mathbf{B}}_2\overline{\boldsymbol{\eta}}_m)\zeta^{m}. \tag{15}$$

The real quantities $\boldsymbol{\Theta\Omega}(\partial\phi^{\infty}/r\partial\theta)$, $\boldsymbol{\Omega}\mathbf{u}^{\infty}$ and $\boldsymbol{\Omega}\phi^{\infty}$ can be expressed in the following forms

$$\boldsymbol{\Theta\Omega}(\partial\phi^{\infty}/r\partial\theta) = 2\,\mathrm{Re}\left\{\sum_{m=1}^{\infty}\boldsymbol{\Phi}^{\infty}_m\zeta^{-m} + \boldsymbol{\Phi}^{\infty}_0\right\}, \tag{16}$$

$$\boldsymbol{\Omega}\mathbf{u}^{\infty} = 2\,\mathrm{Re}\left\{\sum_{m=1}^{\infty}\mathbf{U}^{\infty}_m\zeta^{-m} + \mathbf{U}^{\infty}_0\right\}, \tag{17}$$

$$\boldsymbol{\Omega}\phi^{\infty} = 2\,\mathrm{Re}\left\{\sum_{m=1}^{\infty}\boldsymbol{\pi}^{\infty}_m\zeta^{-m} + \boldsymbol{\pi}^{\infty}_0\right\}, \tag{18}$$

where $\boldsymbol{\Phi}^{\infty}_n$, $\mathbf{U}^{\infty}_n$ and $\boldsymbol{\pi}^{\infty}_n$, n=0~∞, are constants. Consider a new coordinate system x^*_i, i=1,2,3, obtained by rotating x_{i} ,i=1,2,3, about the x_3-axis an angle θ, these two coordinates systems are related by

$$\mathbf{x}^* = \boldsymbol{\Omega}\mathbf{x}, \tag{19}$$

where $\mathbf{x}^* = [x^*_1, x^*_2, x^*_3]^T$ and $\mathbf{x} = [x_1, x_2, x_3]^T$. If the four matrices $\mathbf{A}^*_{(j)}$ and $\mathbf{B}^*_{(j)}$ is in the x^*_i coordinate system, they are related to $\mathbf{A}_{(j)}$, $\mathbf{B}_{(j)}$ by (Ting [8])

$$\mathbf{A}^*_{(j)} = \boldsymbol{\Omega}\mathbf{A}_{(j)};\quad \mathbf{B}^*_{(j)} = \boldsymbol{\Omega}\mathbf{B}_{(j)}. \tag{20}$$

By using (20), eqns (14) and (15) can be equivalently expressed in the x_i^* coordinate system as

$$-\mathbf{\Phi}_m^\infty e^{-im\theta} - \overline{\mathbf{\Phi}}_m^\infty e^{im\theta} + \sum_{m=1}^{\infty}(im/a_0)\mathbf{\Theta}\mathbf{B}_2^*\mathbf{\eta}_m^* e^{-im\theta} - \sum_{m=1}^{\infty}(im/a_0)\mathbf{\Theta}\overline{\mathbf{B}}_2^*\overline{\mathbf{\eta}}_m^* e^{-im\theta}$$

$$= \mathbf{U}_m^\infty e^{-im\theta} + \overline{\mathbf{U}}_m^\infty e^{im\theta} + \sum_{m=1}^{\infty}(\mathbf{A}_2^*\mathbf{\eta}_m^* - \overline{\mathbf{A}}_2^*\overline{\mathbf{\lambda}}_m^* - \mathbf{A}_1^*\mathbf{\lambda}_{-m}^*)e^{-im\theta} + \sum_{m=1}^{\infty}(\overline{\mathbf{A}}_2^*\overline{\mathbf{\eta}}_m^* - \mathbf{A}_2^*\mathbf{\lambda}_m^* - \overline{\mathbf{A}}_1^*\overline{\mathbf{\lambda}}_{-m}^*)e^{im\theta}$$

$$- \mathbf{A}_1^*\mathbf{\lambda}_0^* - \overline{\mathbf{A}}_1^*\overline{\mathbf{\lambda}}_0^* + \mathbf{A}_1^*\mathbf{\eta}_0^* + \overline{\mathbf{A}}_1^*\overline{\mathbf{\eta}}_0^* . \tag{21}$$

$$\mathbf{\pi}_m^\infty e^{-im\theta} + \overline{\mathbf{\pi}}_m^\infty e^{im\theta} + \sum_{m=1}^{\infty}(\mathbf{B}_2^*\mathbf{\eta}_m^* - \mathbf{B}_1^*\mathbf{\lambda}_{-m}^* - \overline{\mathbf{B}}_1^*\overline{\mathbf{\lambda}}_m^*)e^{-im\theta} + \mathbf{B}_2^*\mathbf{\eta}_0^* + \overline{\mathbf{B}}_2^*\overline{\mathbf{\eta}}_0^* - \mathbf{B}_1^*\mathbf{\lambda}_0^* - \overline{\mathbf{B}}_1^*\overline{\mathbf{\lambda}}_0^*$$

$$= \sum_{m=1}^{\infty}(\mathbf{B}_1^*\mathbf{\lambda}_m^* + \overline{\mathbf{B}}_1^*\overline{\mathbf{\lambda}}_{-m}^* - \overline{\mathbf{B}}_2^*\overline{\mathbf{\eta}}_m^*)e^{im\theta}. \tag{22}$$

Eqns (21) and (22) imply the following two sets of equations

$$\mathbf{U}_m^\infty + \mathbf{\Phi}_m^\infty + (\mathbf{A}_2^* - im/a_0\mathbf{\Theta}\mathbf{B}_2^*)\mathbf{\eta}_m^* - \mathbf{A}_1^*\mathbf{\lambda}_m^* - \mathbf{A}_1\mathbf{\lambda}_{-m}^* = \mathbf{0},\ m = 1,2,... \tag{23}$$

$$\mathbf{\pi}_m^\infty + \mathbf{B}_2^*\mathbf{\eta}_m^* - \overline{\mathbf{B}}_1^*\overline{\mathbf{\lambda}}_m^* - \mathbf{B}_1^*\mathbf{\lambda}_{-m}^* = \mathbf{0},\ m = 1,2,... \tag{24}$$

and

$$\mathbf{A}_1^*\mathbf{\lambda}_0^* - \mathbf{A}_2^*\mathbf{\eta}_0^* = \mathbf{\Phi}_0^\infty + \mathbf{U}_0^\infty , \tag{25}$$

$$\mathbf{B}_1^*\mathbf{\lambda}_0^* - \mathbf{B}_2^*\mathbf{\eta}_0^* = \mathbf{\pi}_0^\infty . \tag{26}$$

After solving eqns. (23) ~ (26), we can finally reach the following results

$$\mathbf{\eta}_0 = \mathbf{B}_2^{-1}(\mathbf{A}_1\mathbf{B}_1^{-1} - \mathbf{A}_2\mathbf{B}_2^{-1})^{-1}(\mathbf{\Phi}_0^\infty + \mathbf{U}_0^\infty - \mathbf{A}_1\mathbf{B}_1^{-1}\mathbf{\pi}_0^\infty) , \tag{27}$$

$$\mathbf{\lambda}_0 = \mathbf{B}_1^{-1}[\mathbf{\pi}_0^\infty + (\mathbf{A}_1\mathbf{B}_1^{-1} - \mathbf{A}_1\mathbf{B}_1^{-1})^{-1}(\mathbf{\Phi}_0^\infty + \mathbf{U}_0^\infty - \mathbf{A}_1\mathbf{B}_1^{-1}\mathbf{\pi}_0^\infty)] , \tag{28}$$

$$\mathbf{\eta}_m = \mathbf{B}_2^{-1}(\mathbf{A}_2\mathbf{B}_2^{-1} - im/a_0\mathbf{\Theta} - \mathbf{I})^{-1}(\mathbf{\pi}_m^\infty - \mathbf{U}_m^\infty - \mathbf{\Phi}_m^\infty),\ m = 1,2,.....\,, \tag{29}$$

$$\mathbf{\lambda}_m = (\mathbf{B}_1^T + \mathbf{A}_1^T)\overline{\mathbf{\pi}}_m^\infty + (\mathbf{B}_1^T + \mathbf{I})(\overline{\mathbf{A}}_2\overline{\mathbf{B}}_2^{-1} + im/a_0\mathbf{\Theta} - \mathbf{I})^{-1}(\mathbf{\pi}_m^\infty - \mathbf{U}_m^\infty - \mathbf{\Phi}_m^\infty) .\ m = 1,2,... \tag{30}$$

4 EXAMPLE:

Consider a circular inclusion of radius a_0 existing in an infinite anisotropic elastic plane which is applied at infinity a uniform stress. The three interface parameters are $h_r= h_\theta= h_z= h$. The displacement $\mathbf{u}^\infty$ and the stress function $\mathbf{\phi}^\infty$ for the matrix can be expressed as (Hwu and Ting [9])

$$\mathbf{u}^\infty = x_1\mathbf{\varepsilon}_1^\infty + x_2\mathbf{\varepsilon}_2^\infty , \quad \phi^\infty = x_1\mathbf{t}_2^\infty - x_2\mathbf{t}_1^\infty , \tag{31}$$

in which

$$\mathbf{\varepsilon}_1^\infty = [\varepsilon_{11}^\infty, 0, 2\varepsilon_{13}^\infty]^T ; \quad \mathbf{\varepsilon}_2^\infty = [2\varepsilon_{21}^\infty, \varepsilon_{22}^\infty, 2\varepsilon_{23}^\infty]^T ; \quad \mathbf{t}_1^\infty = [\sigma_{11}^\infty, \sigma_{12}^\infty, \sigma_{13}^\infty]^T ; \quad \mathbf{t}_2^\infty = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty]^T . \tag{32}$$

By using $x_1 = a_0(\zeta + \zeta^{-1})/2$, $x_2 = -a_0 i(\zeta - \zeta^{-1})/2$, we can obtain

$$\mathbf{\Phi}_0^\infty = h\mathbf{d}\overline{\mathbf{v}};\ \mathbf{\Phi}_1^\infty = h\mathbf{e}_3\mathbf{v};\ \mathbf{\Phi}_2^\infty = h\mathbf{d}\overline{\mathbf{v}};\ \mathbf{\Phi}_m^\infty = \mathbf{0} \text{ for } m \ge 3; \tag{33}$$

$$\mathbf{U}_0^\infty = \mathbf{d}\overline{\mathbf{\varepsilon}};\ \mathbf{U}_1^\infty = \mathbf{e}_3\mathbf{\varepsilon};\ \mathbf{U}_2^\infty = \mathbf{d}\mathbf{\varepsilon};\ \mathbf{U}_m^\infty = \mathbf{0} \text{ for } m \ge 3; \tag{34}$$

$$\boldsymbol{\pi}_0^\infty = -ia_0\mathbf{d}\bar{\mathbf{v}};\ \boldsymbol{\pi}_1^\infty = ia_0\mathbf{e}_3\mathbf{v};\ \boldsymbol{\pi}_2^\infty = ia_0\mathbf{d}\mathbf{v};\ \boldsymbol{\pi}_m^\infty = \mathbf{0} \text{ for } m \geq 3; \tag{35}$$

where $\mathbf{v} = -(\mathbf{t}_1^\infty - i\mathbf{t}_2^\infty)/2$ and $\boldsymbol{\varepsilon} = a_0(\boldsymbol{\varepsilon}_1^\infty - i\boldsymbol{\varepsilon}_2^\infty)/2$. Substituting (33) ~ (35) into (27) ~ (30) leads to the expressions for $\boldsymbol{\lambda}_m$ and $\boldsymbol{\eta}_m$.

5 CONCLUSION

The anisotropic circular inclusion problem with imperfect interface is solved in this paper. The explicit, simple and straightforward features of our results reveal that the solution method of our study can also be applicable to those under more complicate loading.

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