DUCTILE FRACTURE:
MATERIAL REMODELLING IN THE PROCESS ZONE

A. DiCarlo

Mathematical Structures of Materials Physics at DiS, Università degli Studi “Roma Tre”, Roma, Italy

ABSTRACT

Two decades ago, Alexander Chudnovsky proposed a physics-based model for the process zone surrounding the tip of a slowly propagating crack, which conspicuously improves on the phenomenological Dugdale-Barenblatt model (Chudnovsky [1,2]). In the following years, that model was successfully applied—under drastic simplifying hypotheses—by Chudnovsky and coworkers (Huang et al. [3], Stojimirovic et al. [4], Chudnovsky et al. [5], Wen et al. [6]). Recent advances in configurational mechanics (DiCarlo & Quiligotti [7], DiCarlo [8,12], DiCarlo et al. [9–11]) make it now possible to develop Chudnovsky's model into a full-fledged field theory, freeing it from all simplistic hypotheses.

1 INTRODUCTION

On organizers' request, this extended abstract is being written ten months in advance with respect to the real thing. Since I intend to present fresh, unpublished results at the Conference, I give the present text the structure of a research project, announcing my results and explaining why and how I am confident to obtain them in due time.

The next section provides a résumé of the theory of bulk growth set forth by DiCarlo & Quiligotti [7]. Being quite long, it is subdivided into three subsections, devoted to kinematics, dynamics and constitutive theory, respectively. Any continuum theory of growth—be it modelled as spread in bulk or concentrated on surfaces—hinges on three key issues: in kinematics, extra degrees of freedom have to be introduced, in order to distinguish growth from deformation; in dynamics, new balance laws have to be provided, apt to govern the evolution of such degrees of freedom; in constitutive theory, well-founded extensions of the basic principles (material indifference and dissipation) have to be conceived, and fit constitutive classes have to be selected and analyzed.

To the best of my knowledge, the theory in DiCarlo & Quiligotti [7] is the only one where the evolution law for *bulk* growth is obtained as a constitutively augmented balance (so achieving—with different and more comprehensive means—what Gurtin [13] had accomplished for *surface* growth). That theory—while constituting the fundamental basis of the present effort—needs to be extended in a nontrivial way to cope with the peculiar phenomena occurring in the process zone. Such extensions are briefly discussed in the short closing section.

2 A CONTINUUM THEORY OF BULK REMODELLING

The growing bodies considered by DiCarlo & Quiligotti [7] are standard Cauchy continua: the only kinematic descriptor ascribed to their points is *place* in ordinary physical space. In the present context, I shall have to add a *tensorial microstructure* and the corresponding dynamical quantities: see Sect. 3. In order to distinguish growth from deformation, *two* evolving configurations are associated with each body element: its *visible* configuration, describing how it is actually placed in space, and its *relaxed* configuration, describing how it "would like" to be placed, that is, how it would be placed if all of its line elements had their (current) relaxed length. The field of relaxed configurations need not be (and usually is not) compatible, not even locally.

This is an old kinematic idea, primarily introduced to distinguish between elastic and viscoplastic strains by Kröner [14] and Lee [15], and much later imported into growth modelling by Rodriguez et al. [16] (see also Taber [17]). The original contribution by DiCarlo and Quiligotti [7] is in dynamics. As summarized in the following, they obtain the evolution law for bulk remodelling as a constitutively augmented *new* balance, the balance of configurational (or remodelling) couples, *independent* of the standard force balance.

2.1 Kinematics

I regard a body as a smooth manifold $\mathcal{B}$ (with boundary $\partial\mathcal{B}$), and call **placement** any smooth embedding

$$p : \mathcal{B} \to \mathcal{E} \tag{1}$$

of the body into the Euclidean place manifold $\mathcal{E}$, whose translation space will be denoted by $\mathrm{V}\mathcal{E}$. Tangent vectors on the body manifold itself are called **line elements**. The set of all line elements attached to a single body-point $b \in \mathcal{B}$ is called the **body element** at b, and denoted $\mathrm{T}_b\mathcal{B}$ (the *tangent space* to $\mathcal{B}$ at b). The union of all body elements is denoted $\mathrm{T}\mathcal{B}$ (the *tangent bundle* of $\mathcal{B}$).

The **body gradient** ∇p of a placement p is a tensor field on $\mathcal{B}$, whose value at any given point b, denoted by $\nabla p|_b$, maps linearly the body element $\mathrm{T}_b\mathcal{B}$ onto $\mathrm{V}\mathcal{E}$. I call **stance** any tensor field of this kind, *be it a gradient or not*. Therefore, a stance is any smooth mapping

$$\mathrm{P} : \mathrm{T}\mathcal{B} \to \mathrm{V}\mathcal{E}\,, \tag{2}$$

such that the restriction $\mathrm{P}|\mathrm{T}_b\mathcal{B}$ is a linear embedding, for all $b \in \mathcal{B}$. If a stance happens to be the gradient of a placement, I say that it is **induced** by that placement: all placement induces a stance, but a general stance is not induced by any placement, not even locally. Growth is the time evolution of the **relaxed stance** $\mathbb{P}$, just as motion is the time evolution of the **actual placement** p.

The **complete motion** of a growing body is a family of pairs $(p, \mathbb{P})$ smoothly parametrized by the *time line* $\mathcal{T}$ (identified with the real line), and the velocity **realized** along that motion at the time $\tau \in \mathcal{T}$ is the pair of fields (a superposed dot denoting time differentiation):

$$(\,\dot{p}(\tau), \dot{\mathbb{P}}(\tau)\mathbb{P}(\tau)^{-1}\,) : \mathcal{B} \to \mathrm{V}\mathcal{E} \times (\mathrm{V}\mathcal{E} \otimes \mathrm{V}\mathcal{E})\,. \tag{3}$$

The linear space of **test velocities** $\mathfrak{T}$, comprising all smooth fields

$$(\mathrm{v}, \mathbb{V}\,) : \mathcal{B} \to \mathrm{V}\mathcal{E} \times (\mathrm{V}\mathcal{E} \otimes \mathrm{V}\mathcal{E})\,, \tag{4}$$

will play a central role in the next subsection. The **visible velocity** of body-points (with physical dimensions length/time) is given by the vector field v, while the *tensor* field $\mathbb{V}$ gives the **growth velocity** of the corresponding body elements (with physical dimensions 1/time).

2.2 Dynamics

A **force** is primarily a continuous linear real-valued *functional* on the space of test velocities, whose *value* is the **working** expended by that force. DiCarlo & Quiligotti [7] assume that the total working expended on any test velocity $(\mathrm{v},\mathbb{V}\,)\in\mathfrak{T}$ admits the following integral representation:

$$\int_{\mathcal{B}} -\,(\,\mathrm{s}\cdot\mathrm{v}+\mathbb{C}\cdot\mathbb{V}+\mathbb{S}\cdot\mathrm{Dv}\,)\;+\int_{\mathcal{B}}(\,\mathrm{b}\cdot\mathrm{v}+\mathbb{B}\cdot\mathbb{V}\,)\;+\int_{\partial\mathcal{B}}\mathrm{t}_{\partial\mathcal{B}}\cdot\mathrm{v}\;, \tag{5}$$

where the integrals are taken with respect to the *relaxed* volume and surface area of body elements, and D denotes the *relaxed* gradient:

$$\mathrm{Dv}:=(\nabla\mathrm{v})\mathbb{P}^{-1}\,. \tag{6}$$

Because of the compound structure of test velocities (4), the force functional splits additively into a **brute force**, dual to v, and a **remodelling force**, dual to $\mathbb{V}$. Another important splitting is between the **inner** working, given by the first bulk integral in eqn (5), and the **outer** working, given by the remaining sum. The **brute self-force** per unit volume s, the **outer brute bulk-force** per unit volume b, and the **brute boundary-force** per unit area $\mathrm{t}_{\partial\mathcal{B}}$ take values in $\mathcal{VE}$; the **remodelling self-couple** per unit volume $\mathbb{C}$, the **brute Piola stress** $\mathbb{S}$ (also a specific couple!), and the **outer remodelling couple** per unit volume $\mathbb{B}$ take values in $\mathcal{VE}\otimes\mathcal{VE}$.

All balance laws are systematically provided by the **principle of null working**: the total working expended on any test velocity should be zero, i.e., the total force should be the null functional. Skipping the balance of brute forces, which is standard, I present here only the **balance of remodelling couples**:

$$-\,\mathbb{C}+\mathbb{B}=0\quad\text{ on }\mathcal{B}\,. \tag{7}$$

2.3 Constitutive theory

My treatment of constitutive issues rests on two pillars (altogether independent of balance): the *principle of material indifference* to change in observer, and the *dissipation principle*. Both of them deliver strict selection rules on the constitutive prescription for the *inner* force. Such *a priori* restrictions do not apply to the outer force, which is regarded as an adjustable control on the process. In the intented application to ductile fracture, $\mathbb{B}$ will be trivial—once the theory is extended to include *thermo*-mechanics.

I will only flash the outcome of the first principle, as applied by DiCarlo & Quiligotti [7], while summarizing with some more detail the machinery of the second one. Material indifference rules out non-trivial values of the brute self-force s and non-symmetric values of the *Cauchy stress* $\mathrm{T}:=|\det\mathrm{F}|^{-1}\,\mathbb{S}\,\mathrm{F}^{\top}$, where the **warp**

$$\mathrm{F}\,:=\,\mathrm{D}p=(\nabla p)\,\mathbb{P}^{-1}. \tag{8}$$

measures how the actual stance, i.e., the body gradient of the actual placement, differs from the relaxed stance. If we further assume that the response of the body element at b filters off from $(p,\mathbb{P})$ all information other than $p|_b$, $\nabla p|_b$, and $\mathbb{P}|_b$, we obtain from the same principle that

$$\mathbb{S}(b,\tau)=\mathrm{R}(b,\tau)\,\check{\mathbb{S}}_b(\mathrm{U}|_b,\mathbb{P}|_b,\tau)\,,\qquad \mathbb{C}(b,\tau)=\,\check{\mathbb{C}}_b(\mathrm{U}|_b,\mathbb{P}|_b,\tau)\,, \tag{9}$$

the *rotation* R and the *stretch* U being, respectively, the orthogonal and the right positive-symmetric factor of the warp (8): $\mathrm{F} = \mathrm{R}\,\mathrm{U}$.

To have a notion of dissipation, an additional energetic descriptor is needed. I postulate the existence of an additive real-valued *free energy* $\Psi(\mathcal{P}) = \int_{\mathcal{P}} \psi$, measuring the inner energy available to body-parts. I call *power expended* along a process at time τ the *opposite* of the working expended by the inner force due to the process on the velocity *realized* at time τ. Hence, the power expended measures the working done by an *outer* force *balanced* with the *constitutively determined* inner force. The **dissipation principle** I enforce requires that the rate of energy dissipation—defined as the difference between the power expended along a process and the time derivative of the free energy—should be non-negative, for all body-parts, at all times. This localizes into:

$$\dot{\psi} \leq \mathbb{S} \cdot \mathrm{Dv} + \mathbb{C} \cdot \mathbb{V}\,, \tag{10}$$

it being intended that v and $\mathbb{V}$ are given by eqn (3), $\mathbb{S}$ and $\mathbb{C}$ by eqn (9), and ψ has to be related to the process by an extra constitutive mapping. The **main constitutive assumption** of DiCarlo & Quiligotti [7] selects a special constitutive class, beautifully accounted for by Epstein [18]. This constitutive class, however interesting and worth consideration, is too narrow for the intented application: here is another point where the theory needs being generalized—incorporating ideas from DiCarlo et al. [10,11]—, to account for *progressive damage* in the process zone. In fact, Epstein [18] and DiCarlo & Quiligotti [7] posit that, at each body-point, the present value of the free energy per unit *relaxed* volume depends *only* on the present value of the *warp* at that point:

$$\psi(b,\tau) = \check{\psi}_b\left(\mathrm{F}(b,\tau)\right). \tag{11}$$

The requirement that inequality (10) be satisfied along all processes is fulfilled if and only if for each b (which will be dropped from now on) the responses $\check{\mathbb{S}}$ and $\check{\mathbb{C}}$ satisfy (∂ denotes differentiation and I the identity on $\mathrm{V}\mathcal{E}$):

$$\check{\mathbb{S}} = \partial\check{\psi} + \overset{+}{\mathbb{S}}\,, \qquad \check{\mathbb{C}} = \mathbb{E} + \overset{+}{\mathbb{C}}\,, \tag{12}$$

with

$$\mathbb{E} := \check{\psi}\,\mathrm{I} - \mathrm{F}^{\top}\check{\mathbb{S}} \tag{13}$$

the **Eshelby coupling** between brute mechanics and remodelling, and the *extra-energetic responses* $\overset{+}{\mathbb{S}}, \overset{+}{\mathbb{C}}$ restricted by the **reduced dissipation inequality**

$$\overset{+}{\mathbb{S}} \cdot \dot{\mathrm{F}} + \overset{+}{\mathbb{C}} \cdot \mathbb{V} \geq 0\,, \tag{14}$$

to be abided by in the same sense as inequality (10). As is seen, the Eshelbian coupling (13) is mandatory, within the constitutive class considered, and independent of any special assumption on $\check{\mathbb{S}}$. Additional constitutive couplings are not ruled out, of course.

3 HOW TO MODEL CAVITATION AND CRAZING

A process zone, i.e., a zone of large deformation in the vicinity of the crack tip is observed above the transition temperature from brittle to ductile mode of fracture. A microscopic examination reveals that the process zone consists of dense cavitations and crazes that form stripes, well visible on the millimeter scale. At a higher magnification, in the 10–100 μm range, each individual stripe appears as a set of well aligned and connected cavities, that a coarse-grained theory would describe collectively as a displacement discontinuity. On the micrometer scale, a sponge-like structure of the cavities is observable with a high resolution scanning electron microscope. A root cause of cavitation and crazing is *micronecking*, which constitutes the major energy absorption mechanism within the process zone (Chudnovsky & Preston [19]).

In the continuum theory I contemplate, the evolving distribution of cavitations and crazes is modelled by endowing the body with a rich *microstructure*, consisting of two *tensor* fields, V and A, representing **voids** per unit volume (Harrigan and Mann [20], Cowin et al. [21], Aubouy et al. [22]) and **void boundaries** per unit volume (Wetzel and Tucker [23]), respectively. The complete motion of the body and the corresponding velocities (3), (4) are extended accordingly. In dynamics, the balance of remodelling couples (7) is supplemented by two further *micro-configurational* balances, involving **crazing couples**, dual to V and A, respectively. With respect to eqn (11), the constitutive class under consideration is extended assuming that

$$\psi(b,\tau) = \check{\psi}_b\left(\mathrm{F}(b,\tau), \mathsf{V}(b,\tau), \mathsf{A}(b,\tau)\right), \tag{15}$$

i.e., at each body-point, the present value of the free energy per unit *relaxed* volume $\psi(b,\tau)$ depends not only on the present value of the *warp* F, but also on the present values of the *void* tensor V and of the *void-boundary* tensor A at that point. In this constitutive class, which is much larger than that defined by eqn (11), the Eshelbian coupling includes— but is much richer than—that provided by eqn (13).

REFERENCES

1. Chudnovsky A., *The Crack Layer Theory*, NASA Contractor Report No. 174634 (1984).
2. Chudnovsky A., Crack layer theory. In: Proc. 10th U.S. Nat. Conf. Appl. Mech., Austin, TX (Lamb J.P., ed.), ASME, 97–106 (1986).
3. Huang W.-L., Kunin B., and Chudnovsky A., Kinematics of damage zone accompanying curved crack. Int. J. Fracture **50**, 143–152 (1991).
4. Stojimirovic A., Kadota K., and Chudnovsky A., An equilibrial process zone in polymeric materials. J. Appl. Polymer Science **46**, 1051–1056 (1992).
5. Chudnovsky A. and Shulkin Y., New model of slow crack growth in viscoelastic thermoplastics. In: Proc. Conf. SPE/ANTEC '96, Indianapolis, IN, Society of Plastics Engineers, **3**, 3284 (1996).
6. Wen Z., Gorelik M., Chudnovsky A., Dudley J.W., and Shlyapobersky J., Observation and characterization of crack growth in porous rocks. In: *Rock Mechanics, Tools and Techniques* (M. Aubertin, ed.), 1269–1277 (1996).
7. DiCarlo A. and Quiligotti S., Growth and balance, Mech. Res. Comm. **29**, 449–456 (2002).
8. DiCarlo A., Surface and bulk growth unified. In: *Mechanics of Material Forces* (P. Steinmann and G.A. Maugin, eds.), Kluwer, Dordrecht (in press).
9. DiCarlo A., Nardinocchi P., and Teresi L., Creep as passive growth (forthcoming).
10. DiCarlo A., Naili S., and Quiligotti S., Remodelage rotatoire de la raideur élastique (forthcoming).

11. DiCarlo A., Naili S., and Quiligotti S., Rotary remodelling of elasticity tensors in the plane (forthcoming).

12. DiCarlo A., *Material Remodelling* (in preparation).

13. Gurtin M.E., *Configurational Forces as Basic Concepts of Continuum Physics*, Springer-Verlag, Berlin (2000).

14. Kröner E., Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen, Arch. Rational Mech. Anal. **4**, 273–334 (1960).

15. Lee E.H., Elastic-plastic deformations at finite strains, J. Appl. Mechanics **36**, 1–6 (1969).

16. Rodriguez E.K., Hoger A., and McCulloch A.D., Stress-dependent finite growth in soft elastic tissues, J. Biomechanics **27**, 455–467 (1994).

17. Taber L.A., Biomechanics of growth, remodeling, and morphogenesis, Appl. Mech. Rev. **48**, 487–545 (1995).

18. Epstein M., On material evolution laws. In: *Geometry, Continua & Microstructure* (G.A. Maugin, ed.), Hermann, Paris, 1–9 (1999).

19. Chudnovsky A. and Preston S., Configurational mechanics of necking phenomena in engineering thermoplastics. Mech. Res. Comm. **29**, 465–475 (2002).

20. Harrigan T.P. and Mann R.W., Characterization of microstructural anisotropy in orthotropic materials using a second rank tensor. J. Mater. Sci. **19**, 761–767 (1984).

21. Cowin S.C., Sadegh A.M., and Luo G.M., An evolutionary Wolff's law for trabecular architecture. J. Biomech. Eng. **114**, 129–136 (1992).

22. Aubouy M., Jiang Y., Glazier J.A., and Graner F., A texture tensor to quantify deformations. Granular Matter **5**, 67–70 (2003).

23. Wetzel E.D. and Tucker C.L. III, Area tensors for modeling microstructure during laminar liquid-liquid mixing. Int. J. Multiphase Flow **25**, 35–61 (1999).