REGULARIZED DAMAGE MODEL BASED ON NONLOCAL DISPLACEMENT FIELD

S. Marfia[1] and M. Jirásek[2]

[1] Department of Mechanics, Structures and Environment, University of Cassino, Italy

[2] Department of Structural Mechanics, Czech Technical University in Prague, Czech Republic

ABSTRACT

Continuum damage models describe the changes of material stiffness and strength, caused by the evolution of defects, in the framework of continuum mechanics. In many materials, a fast evolution of defects leads to stress-strain laws with softening, which creates serious mathematical and numerical problems. To regularize the model behavior, various generalized continuum theories have been proposed. Integral-type nonlocal damage models are usually based on weighted spatial averaging of a strain-like quantity. This paper explores an alternative formulation with averaging of the displacement field. It is shown that an exact equivalence between strain and displacement averaging can be achieved only in an unbounded medium. Around physical boundaries of the analyzed body, both formulations differ and the nonlocal displacement model generates spurious damage in the boundary layers. The paper shows that this undesirable effect can be surpressed by an appropriate adjustment of the nonlocal weight function. The first numerical tests indicate that, in terms of global characteristics such as the load-displacement curves, the results obtained with the newly proposed formulation are very similar to those obtained with the usual formulation based on strain averaging, but locally they give a smoother distribution of stress and surpress oscillations observed for the usual formulation in certain regions of the process zone.

1 INTRODUCTION

Continuum damage mechanics is a constitutive theory that describes the progressive loss of material integrity due to the propagation and coalescence of microcracks, microvoids, and similar defects. These changes in the microstructure lead to a degradation of material stiffness observed on the macroscale. The density and orientation of microdefects can be approximated by certain internal variables whose number and character depend on the complexity of the model.

In many structures subjected to extreme loading conditions, the initially smooth distribution of strain changes into a highly localized one. Typically, the strain increments are concentrated in narrow zones of evolving microdefects while the major part of the structure experiences unloading. Standard “local” damage models fail to describe localized failure patterns in an objective way. A possible remedy consists in reformulating the constitutive model as nonlocal. Integral-type nonlocal models abandon the classical assumption of locality and admit that stress at a certain point can depend not only on the state variables at that point but in general on the distribution of state variables over the whole body, or at least on their distribution in a finite neighborhood of the point under consideration. The first models of this type, proposed in the 1960s, aimed at improving the description of elastic wave dispersion in crystals. Nonlocal elasticity was further developed by Eringen, and in the early 1980s it was extended to nonlocal elastoplasticity. Subsequently it was found that certain nonlocal formulations can act as efficient localization limiters with a regularizing effect on problems with damage localization (Pijaudier-Cabot and Bažant [1]).

2 NONLOCAL DAMAGE MODELS WITH STRAIN AVERAGING

The typical format of the stress-strain law used by damage models reads

$$\boldsymbol{\sigma} = \boldsymbol{D}_s(\boldsymbol{\Omega})\,\boldsymbol{\varepsilon} \tag{1}$$

where $\boldsymbol{\sigma}$ is the column matrix of stress components, $\boldsymbol{\varepsilon}$ is the column matrix of engineering strain components, and $\boldsymbol{D}_s$ is the damaged (secant) material stiffness matrix, which depends on certain damage variables collectively denoted by the symbol $\boldsymbol{\Omega}$. For instance, for the simple isotropic damage model, $\boldsymbol{\Omega}$ reduces to a scalar, ω, and the secant stiffness is written as $\boldsymbol{D}_s(\omega) = (1-\omega)\boldsymbol{D}_e$ where $\boldsymbol{D}_e$ is the elastic stiffness of the undamaged material. The growth of the damage variable(s) must be described by a suitable evolution equation.

Generally speaking, the nonlocal approach consists in replacing a certain variable by its nonlocal counterpart obtained by weighted averaging over a spatial neighborhood of each point under consideration. The choice of the variable to be averaged is often guided by intuition, and the localization properties of each specific formulation must be checked carefully.

A widely used nonlocal isotropic damage model that gives a satisfactory description of the entire localization process up to complete failure is based on averaging of the so-called equivalent strain, which is a scalar measure of the strain level driving the evolution of damage. A natural extension of this nonlocal formulation to anisotropic damage models consists in replacing the strain $\boldsymbol{\varepsilon}$ in the evolution equations (but not in the stress-strain law) by the nonlocal strain

$$\overline{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{\varepsilon}(\boldsymbol{\xi})\;\mathrm{d}\boldsymbol{\xi} \tag{2}$$

Here, V is the body of interest, and $\alpha(\boldsymbol{x},\boldsymbol{\xi})$ is a given nonlocal weight function. In an infinite isotropic body, the weight function is assumed to depend only on the distance between the “source” point, $\boldsymbol{\xi}$, and the “target” point, $\boldsymbol{x}$. In the vicinity of a boundary, the weight function is usually rescaled such that the nonlocal operator does not alter a uniform field, in other words, such that the weight function satisfies the normalizing condition

$$\int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})\;\mathrm{d}\boldsymbol{\xi} = 1 \qquad \text{for all } \boldsymbol{x}\in V \tag{3}$$

This can be achieved by setting

$$\alpha(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\alpha_0(||\boldsymbol{x}-\boldsymbol{\xi}||)}{\int_V \alpha_0(||\boldsymbol{x}-\boldsymbol{\zeta}||)\;\mathrm{d}\boldsymbol{\zeta}} \tag{4}$$

where $\alpha_0(r)$ is a monotonically decreasing nonnegative function of the distance $r = ||\boldsymbol{x}-\boldsymbol{\xi}||$. The weight function is often taken as the truncated quartic polynomial function

$$\alpha_0(r) = \begin{cases} \left(1-\dfrac{r^2}{R^2}\right)^2 & \text{if } 0 \le r \le R \\ 0 & \text{if } R \le r \end{cases} \tag{5}$$

where R is a parameter related to the internal length of the material, which is dictated by the size and spacing of major heterogeneities. Since R corresponds to the largest distance of point $\boldsymbol{\xi}$ that affects the nonlocal average at point $\boldsymbol{x}$, it is called the interaction radius.

The nonlocal formulation with damage driven by nonlocal strain was successfully used e.g. for the rotating crack model (Jirásek and Zimmermann [2]), which can be interpreted as a special type of anisotropic damage model.

3 NONLOCAL DAMAGE MODEL WITH DISPLACEMENT AVERAGING

It is interesting to explore what could be gained by a modified formulation with averaging of the displacement field. The idea of a damage model with nonlocal displacements was first mentioned by Huerta, Rodríguez-Ferran and Morata [3], but only as a general concept, without further details or numerical examples. As will be shown here, the actual implementation of this approach in a general multi-dimensional context is not trivial, because it requires certain modifications of the averaging operator.

As a starting point, we will analyze the relation between nonlocal strains and nonlocal displacements. One can expect that nonlocal strains are closely related to the symmetric gradient of nonlocal displacements. Integrating by parts the nonlocal average of the displacement gradient, we obtain

$$\int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})\frac{\partial u_i(\boldsymbol{\xi})}{\partial \xi_j}\,\mathrm{d}V(\boldsymbol{\xi}) = \int_S \alpha(\boldsymbol{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi})n_j(\boldsymbol{\xi})\,\mathrm{d}S(\boldsymbol{\xi}) - \int_V \frac{\partial \alpha(\boldsymbol{x},\boldsymbol{\xi})}{\partial \xi_j}u_i(\boldsymbol{\xi})\,\mathrm{d}V(\boldsymbol{\xi}) \tag{6}$$

In the special case when the nonlocal weight function depends only on the distance between the source and the target, i.e.,

$$\alpha(\boldsymbol{x},\boldsymbol{\xi}) = c_0\alpha_0(\|\boldsymbol{x}-\boldsymbol{\xi}\|) \tag{7}$$

where c_0 is a constant scaling factor and $\alpha_0(r)$ is an even, continuously differentiable function, we have

$$\frac{\partial \alpha}{\partial \boldsymbol{x}} = -\frac{\partial \alpha}{\partial \boldsymbol{\xi}} \tag{8}$$

and (6) can be rewritten as

$$\int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})\frac{\partial u_i(\boldsymbol{\xi})}{\partial \xi_j}\,\mathrm{d}V(\boldsymbol{\xi}) = \int_S \alpha(\boldsymbol{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi})n_j(\boldsymbol{\xi})\,\mathrm{d}S(\boldsymbol{\xi}) + \frac{\partial}{\partial x_j}\int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi})\,\mathrm{d}V(\boldsymbol{\xi}) \tag{9}$$

If the distance of point $\boldsymbol{x}$ from the boundary S is larger that the interaction distance R, the weight function $\alpha(\boldsymbol{x},\boldsymbol{\xi})$ vanishes for $\boldsymbol{\xi} \in S$ and the surface integral in (9) disappears. Equation (9) then means that the nonlocal average of the displacement gradient, $\overline{u_{i,j}}$, is equal to the gradient of the nonlocal displacement, $\overline{u_i}_{,j}$. Since strain is the symmetric part of the displacement gradient (in the small-strain theory used here), the nonlocal strain is indeed related to the nonlocal displacement by the same kinematic equations that link the local strain to the local displacement:

$$\overline{\varepsilon_{ij}} = \frac{1}{2}\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) \tag{10}$$

But this elegant relation holds only in the interior part of the body, not influenced by boundary effects. In boundary layers of thickness R, the averaging function usually does not have the form (7), but even if it had, the surface integral in (9) would not vanish, and so relation (10) would not be valid.

Instead of trying to construct a model exactly equivalent to the traditional one, we will explore the potential of an alternative formulation, accepting that it will not give exactly the same results for finite bodies. The main idea is that the evolution of damage is driven by the symmetric gradient of the nonlocal displacement field. At a sufficient distance from the boundary, this quantity is equal to the nonlocal strain, so the modification of the model concerns only the boundary layers.

To test the proposed model with displacement averaging, let us simulate bending of a simply supported unnotched beam loaded by a force at the axis of symmetry using a structured finite element mesh consisting of 15×60 bilinear quadrilaterals. The nonlocal isotropic damage model based on

Figure 1: Distribution of damage in the three-point bending test obtained by nonlocal damage models with strain averaging and with displacement averaging (using the standard weight function)

averaging of equivalent strain gives reasonable results with damage concentrated in a process zone around the axis of symmetry (Fig. 1 left). However, for the model based on displacement averaging, damage appears also all along the boundaries (Fig. 1 right). This is certainly a pathological effect.

The reason for the poor performance of the nonlocal displacement formulation is that the variable driving damage is not invariant with respect to rigid-body motions. The weight function (4) is constructed such that the averaging operator preserves a constant field. This is sufficient for the model with strain averaging, because rigid-body motions lead to vanishing (therefore, constant) local strain and the corresponding nonlocal strain vanishes as well. If the rigid-body motion is restricted to a translation, the displacement field is constant, its nonlocal average is also constant, and the gradient of nonlocal displacements vanishes, so that damage does not grow. However, rigid-body rotation is associated with a displacement field that varies as a linear function of the spatial coordinates, and the nonlinear operator produces a field that is more general and its symmetric gradient does not vanish. This is inadmissible, because pure rigid-body rotation would induce damage in boundary layers of thickness R. In the bending test, even if the specimen does not rotate globally, local rotation still leads to spurious damage along the boundaries.

Fortunately, the pathological effects can be removed by an appropriate extension of the normalizing condition (3). In the new approach based on displacement averaging, it is not sufficient to ensure invariance of a constant field with respect to the nonlocal operator. This would only mean that uniform translation of the body leads to a uniform nonlocal displacement field and, consequently, to a vanishing symmetric gradient of nonlocal displacement. The simplest way to extend the invariance condition to all rigid-body motions is to require invariance of linear displacement fields. This condition at the same time ensures that if the local strain field is constant, then the symmetric gradient of nonlocal displacement will be constant as well.

The foregoing considerations lead to the extended normalizing conditions

$$\int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = 1 \tag{11}$$

$$\int_V (\boldsymbol{\xi}-\boldsymbol{x})\alpha(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \boldsymbol{0} \tag{12}$$

which should be satisfied at every point $\boldsymbol{x} \in V$. For points sufficiently far from the boundary, the weight function in the form (7) satisfies conditions (12) automatically (because of symmetry), and condition (11) is the same as in the standard case and can be easily satisfied by proper scaling of the weight function, using a constant scaling factor c_0. However, in the boundary layers it is not sufficient to use the standard scaling procedure with c_0 replaced by $c_0(\boldsymbol{x})$, because the resulting function would satisfy only condition (11) but not conditions (12). The procedure proposed here is to multiply the weight function not by a single scaling factor independent of $\boldsymbol{\xi}$ but by a linear function of $\boldsymbol{\xi}$, with coefficients dependent on $\boldsymbol{x}$. So the proposed expression for the modified weight function has the form

$$\alpha(\boldsymbol{x}, \boldsymbol{\xi}) = [p_0(\boldsymbol{x}) + (\boldsymbol{\xi} - \boldsymbol{x})^T \boldsymbol{p}_1(\boldsymbol{x})]\alpha_0(\|\boldsymbol{x} - \boldsymbol{\xi}\|) \tag{13}$$

where p_0 and $\boldsymbol{p}_1$ are unknowns to be determined from the normalizing conditions (the size of column matrix $\boldsymbol{p}_1$ corresponds to the number of spatial dimensions). Substituting (13) into (11)–(12) and rearranging the terms, we obtain a set of linear equations for p_0 and $\boldsymbol{p}_1$. It can be shown that, under certain mild assumptions, the system matrix is always regular, and the modified weight function (13) can be uniquely determined.

4 EXAMPLE

The performance of the nonlocal damage model with displacement averaging using the modified averaging scheme was tested on several examples. First, it was checked that no spurious damage is generated by rigid rotations. As expected, this test was passed without any problems, which proved that the proposed modification of the weight function can indeed improve the results. Next, the three-point bending test was simulated with the newly proposed model. As shown in the right part of Fig. 2a, the distribution of damage is now reasonable and no spurious damage is generated along the boundary (cf. Fig. 1d showing the spurious damage for the model with displacement averaging using the standard nonlocal weight function).

The distribution of damage is quite similar for the models with strain averaging and with displacement averaging. However, interesting differences can be found for the stresses. The longitudinal stress (component σ_{xx}) is plotted in of Fig. 2b. The overall stress distribution is similar but the model with strain averaging leads to stress oscillations in the process zone, which are not seen for the model with displacement averaging.

Fig. 3a shows that the load-displacement curves obtained with both models (NLS = nonlocal strain, NLD = nonlocal displacement) are almost identical, not only for the quadrilateral mesh (Q4) from Fig. 1b, but also for a mesh composed of constant-strain triangles (T3).

Additional tests will be presented at the conference and reported in a full journal version of this paper.

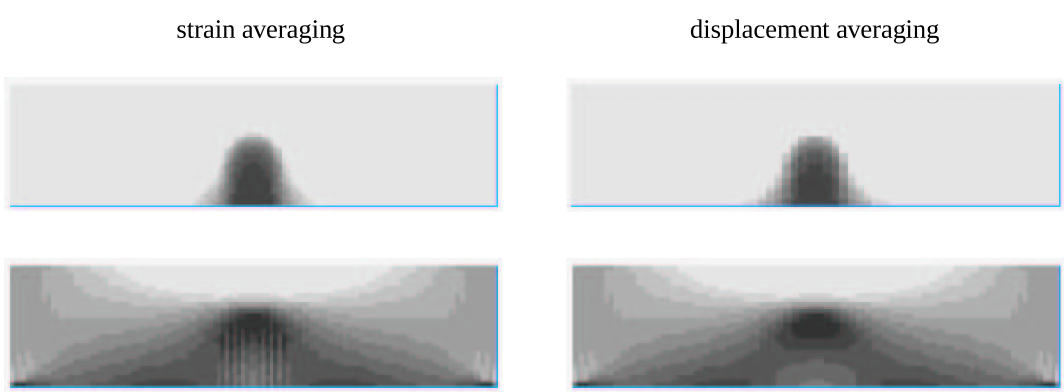

Figure 2: Distribution of damage (top) and stress σ_{xx} (bottom) in the three-point bending test obtained by nonlocal models with strain averaging and with displacement averaging (using the modified weight function)

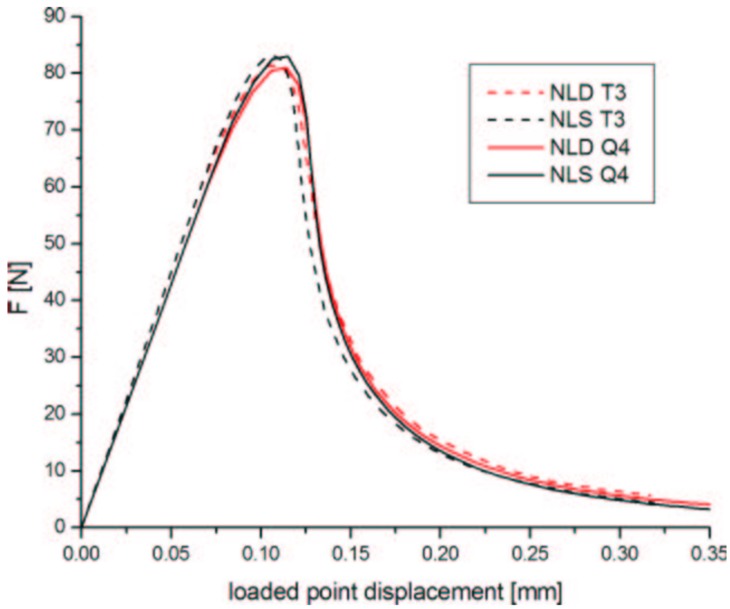

Figure 3: Load-displacement curves for different models and meshes (NLD = nonlocal displacement, NLS = nonlocal strain, Q4 = quadrilateral mesh, T3 = triangular mesh)

5 DISCUSSION AND CONCLUSIONS

Numerical scheme working with the interpolation of nonlocal displacements based on their nodal values provides a better balance between individual terms that enter the stress evaluation formula. Stress at a certain Gauss point is computed as the product of the local strain and the secant stiffness evaluated from the symmetric gradient of nonlocal displacements. Both the local strain and the symmetric gradient of nonlocal displacements are evaluated from the nodal values of the (local or nonlocal) displacement using the same B-matrix. For instance, in a rectangular bilinear element aligned with the direction of uniaxial loading, both local strain and symmetric gradient of nonlocal displacement are constant in the element, and so the resulting stress is constant as well. In contrast to that, in the standard scheme using nonlocal strains, the nonlocal strain is variable (provided that the local strain in the neighboring element is different) and this leads to stress oscillations. Such oscillations are reduced with the newly proposed scheme. Moreover, nonlocal averaging based on nodal contributions is highly efficient. The model with nonlocal displacement field therefore represents an attractive alternative to the commonly used model with nonlocal strain.

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