A GENERAL FORMULATION OF THE POTENTIAL ENERGY RELEASE RATE FOR A THREE-DIMENSIONAL HYPERELASTIC BODY CONTAINING A PLANE CRACK

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ABSTRACT

In this paper a general expression of the potential energy release rate $\overline{\boldsymbol{G}}$ for a three-dimensional fracture mechanics problem is supplied. Under the hypothesis of a quasi-static growth phenomenon, the distribution of the vector **a**, i.e. the velocity field of the fracture propagation, is assumed unknown along the crack front. This assumption leads to a general formula of $\overline{\boldsymbol{G}}$ for a three-dimensional hyperelastic body containing a plane crack. Moreover, imposing a stationary condition of $\overline{\boldsymbol{G}}$ with respect to **a**, the analytical problem of the crack front shape evaluation is formulated. A unique solution exists for the problem, which is described by a system of two non-linear equations. Practical applications of the theory can be obtained by the use of finite element analysis results, together with a numerical solution of the two equations in the unknown components of the fracture propagation velocity.

1 INTRODUCTION

Following the same procedure exposed in (Bennati [1]), a more general expression of $\overline{\boldsymbol{G}}$ (the Potential Energy Release Rate) can be assessed for a three-dimensional hyperelastic solid. This expression contains as parameters both the local direction and the intensity of propagation, i.e. the local velocity of crack growth. If only the direction of propagation along the crack front is assumed as parameter, the expression of $\boldsymbol{G}$, the potential energy release rate relevant to a unit thickness of the three-dimensional solid (obtained in a local reference frame), formally does not depend directly on the parameter itself. On the other hand assuming as parameter the vector **a**, the velocity field of the fracture propagation defined along the crack front line, a general expression of $\overline{\boldsymbol{G}}$ which directly depends on **a** and its gradient can be obtained. This expression allows us to impose a stationary condition of $\overline{\boldsymbol{G}}$ with respect to the vector **a** under the simplified hypothesis of a quasi-static propagation of the fracture. In this way a system of two non-linear equations has been defined and can be used to analyse the evolution of the shape of a three-dimensional plane crack propagating in a hyperelastic solid.

2 THREE-DIMESIONAL FORMULATION OF $\boldsymbol{G}$

Considering a three-dimensional cracked body $\mathcal{B}$, the quasi-statically propagating fracture can be represented by means of a regular surface, whose area a_f can be assumed as the time parameter. The crack surface is bounded by the regular curve $\gamma(a_f)$; assuming a global reference frame as shown in Figure 1, the parametric equation of the curve is:

$$\mathbf{X} = \mathbf{Z}(X_3, a_f) \ . \tag{1}$$

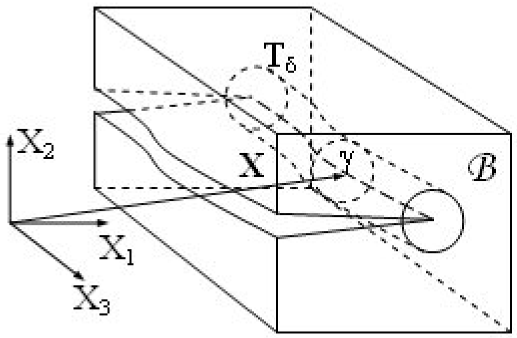

Figure 1.

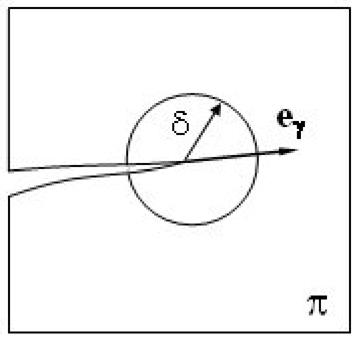

Figure 2.

For each value of the abscissa X_3, the crack tip grows with velocity $\mathbf{a}(X_3, a_f)$ in the direction of the unit vector $\mathbf{e}_\gamma$ defined as:

$$\mathbf{e}_\gamma(X_3, a_f) = \frac{\partial \mathbf{Z}(X_3, a_f)}{\partial a_f} \quad . \tag{2}$$

In the following (Figure 1), T_δ indicates a tubular sub-region of $\mathcal{B}$, whose intersection with the plane π (normal plane) at any point along the crack tip is a circle with centre on γ and radius δ (Figure 2) and $\mathcal{B}_\delta = \mathcal{B} - T_\delta$. We can suppose $\mathbf{e}_\gamma$ lying in the π plane; moreover, when we deal with a plane crack, it will coincide with the normal on γ.

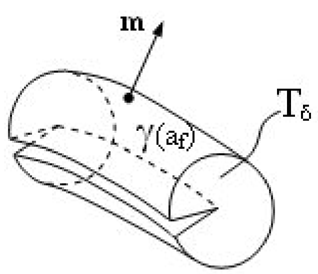

Figure 3.

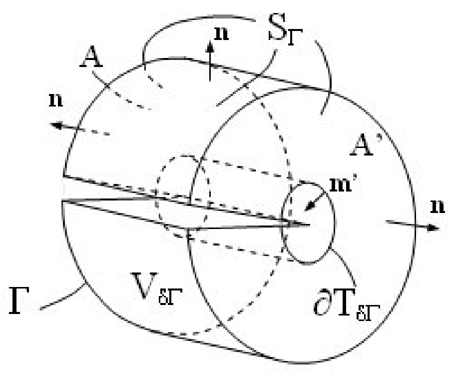

Figure 4.

As demonstrated in (Bennati [1], Rice [2]), the potential energy release rate, $\overline{\boldsymbol{G}}(a_f)$, of the whole body $\mathcal{B}$ for a unit increment of cracked area a_f assumes the following expression:

$$\overline{\boldsymbol{G}}_B(a_f) = \lim_{\delta \to \mathbf{0}} \int_{\partial T_\delta} (\sigma\, \mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}} \nabla \mathbf{u}\, \mathbf{e}_\gamma) \cdot \mathbf{m}\, dS \quad , \tag{3}$$

where σ is the strain energy density, $\mathbf{u}$ the displacement vector field, T the Cauchy stress tensor and $\mathbf{m}$ is the unit vector normal to ∂T_δ (Figure 3). In the present case, using tensor algebra calculations (Gurtin [3]), the divergence of the integral function in the eqn (3) can be written as:

$$\operatorname{div}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma) = \sigma\operatorname{div}\mathbf{e}_\gamma + \operatorname{div}(\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\mathbf{e}_\gamma - (\nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\nabla\mathbf{e}_\gamma\,. \tag{4}$$

Since

$$\sigma\operatorname{div}\mathbf{e}_\gamma = \sigma\frac{\mathrm{d}e_{\gamma 3}}{\mathrm{d}X_3} = \sigma\,\mathrm{I}\bullet\nabla\mathbf{e}_\gamma \tag{5}$$

we obtain (I is the identity tensor):

$$\operatorname{div}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma) = (\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\nabla\mathbf{e}_\gamma + \operatorname{div}(\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\mathbf{e}_\gamma\ . \tag{6}$$

Applying the divergence theorem to a volume $V_{\delta\Gamma}$, whose boundaries are the regular surface S_Γ, the crack faces S_{f} and the boundary $\partial T_{\delta\Gamma}$ with outward unit normal vectors **n** and **m'=-m** respectively (Figure 4), we have:

$$\int_{V_{\delta\Gamma}}\operatorname{div}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\,dV =$$

$$= \int_{S_\Gamma + S_{\mathrm{f}}}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\bullet\mathbf{n}\,dS + \int_{\partial T_{\delta\Gamma}}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\bullet\mathbf{m}'\,dS\quad ; \tag{7}$$

moreover, since in the present case $\operatorname{div}(\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T}) = \mathbf{0}$ (Eshelby [4]), in view of (6)

$$\int_{\partial T_{\delta\Gamma}}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\bullet\mathbf{m}\,dS =$$

$$= \int_{S_\Gamma + S_{\mathrm{f}}}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\bullet\mathbf{n}\,dS - \int_{V_{\delta\Gamma}}(\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\nabla\mathbf{e}_\gamma\,dV\quad , \tag{8}$$

Finally, in the case of a plane crack (i.e. $\mathbf{n}\cdot\mathbf{e}_\gamma = 0$ on S_{f}), we obtain:

$$\int_{\partial T_{\delta\Gamma}}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\bullet\mathbf{m}\,dS = \int_{S_\Gamma}(\sigma\,\mathbf{e}_\gamma - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u}\,\mathbf{e}_\gamma)\bullet\mathbf{n}\,dS - \int_{V_{\delta\Gamma}}(\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\nabla\mathbf{e}_\gamma\,dV\ , \tag{9}$$

being $\mathbf{e}_\gamma$ not a constant vector field. Therefore, for each volume V_Γ,

$$\overline{\boldsymbol{G}}_{V_\Gamma}(\mathrm{a}_{\mathrm{f}}) = \int_{S_\Gamma}(\sigma\,\mathrm{I} - \mathrm{T}^{\mathrm{T}}\nabla\mathbf{u})\,\mathbf{e}_\gamma\bullet\ \mathbf{n}\,dS\ - \int_{V_\Gamma}(\sigma\,\mathrm{I} - \nabla\mathbf{u}^{\mathrm{T}}\mathrm{T})\bullet\nabla\mathbf{e}_\gamma\,dV\ . \tag{10}$$

From eqn (10) we can obtain the expression of $\boldsymbol{G}(\mathrm{X}_{\mathbf{3}},\mathrm{a}_{\mathrm{f}})$, which is the local value of the potential energy release rate along the crack front. In fact, if the control volume V_Γ is a cylinder with height dl:

$$\boldsymbol{G}(\mathrm{X}_{\mathbf{3}},\mathrm{a}_{\mathrm{f}})\,dl =$$

$$= dl\ \mathrm{e}_{\gamma k}\int_{\Gamma}(\sigma\,\mathrm{n}_{\mathrm{k}} - \mathrm{t}_{\mathrm{i}}\mathrm{u}_{\mathrm{i,k}})\,ds + \mathrm{e}_{\gamma k}\int_{A+A'}(\sigma\,\mathrm{n}_{\mathrm{k}} - \mathrm{t}_{\mathrm{i}}\mathrm{u}_{\mathrm{i,k}})\,dS - dl\ \mathrm{e}_{\gamma k,\mathbf{3}}\int_{A}(\sigma\,\mathrm{n}_{\mathrm{k}} - \mathrm{t}_{\mathrm{i}}\mathrm{u}_{\mathrm{i,k}})\,dS\ , \tag{11}$$

where a local reference frame has been assumed, with the x_1-x_3 plane parallel to the crack surface and the x_1-x_2 plane, which contains the base A, coinciding with the normal one.

As stated before, the local direction of crack propagation $\mathbf{e}_\gamma$ coincides with the normal to the crack front and the global abscissa X_3 locates the origin of the local reference frame. The third integral in the eqn (11) has been obtained averaging the function $(\sigma\,\mathrm{I} - \nabla \mathbf{u}^T \mathrm{T}) \bullet \nabla \mathbf{e}_\gamma$ on the length *dl*. Since the normal vectors are opposite on the bases *A* and *A'*, using the Kronecker delta:

$$e_{\gamma k} \int_{A'} (\sigma\, n_k - t_i u_{i,k})\, dS =$$

$$= - e_{\gamma k} \int_A (\sigma\, n_k - t_i u_{i,k})\, dS - dl\, e_{\gamma k} \int_A (\sigma\, n_k - t_i u_{i,k})_{,3}\, dS - dl\, e_{\gamma k,3} \int_A (\sigma\, n_k - t_i u_{i,k})\, dS \,, \quad (12)$$

moreover, being $\mathbf{n} \equiv -\mathbf{i}_3$ on *A*, $\sigma\, n_k = -\sigma\, \delta_{k3}$ and $t_i = -T_{ij}\, n_j = -T_{i3}$, then:

$$\boldsymbol{G}(X_3, a_f)\, dl = dl\; e_{\gamma k} \int_\Gamma (\sigma\, n_k - t_i u_{i,k})\, ds + dl\; e_{\gamma k} \int_A (\sigma\, \delta_{k3} - T_{i3} u_{i,k})_{,3}\, dS +$$

$$+ dl\; e_{\gamma k,3} \int_A (\sigma\, \delta_{k3} - T_{i3} u_{i,k})\, dS - dl\; e_{\gamma k,3} \int_A (\sigma\, \delta_{k3} - T_{i3} u_{i,k})\, dS \;. \quad (13)$$

Being $e_{\gamma 1} = 1$, $e_{\gamma 2} = e_{\gamma 3} = 0$ in the local reference frame,

$$\boldsymbol{G}(X_3, a_f) = \int_\Gamma (\sigma\, n_1 - t_i u_{i,1})\, ds - \int_A (T_{i3} u_{i,1})_{,3}\, dS \;. \quad (14)$$

The previous relation is well known in literature and it has been used in (Chiarelli [5]). This is the local energy release rate for a unit increment of the cracked area a_f, obtained under the assumption of a constant local rate of propagation along a known direction (normal to the crack front line).

3 ANALYSIS OF THREE-DIMESIONAL CRACK FRONT SHAPE

The aim of this section is the assessment of an analytical formulation of the three-dimensional crack front shape for a quasi-static propagation in an hyperelastic media. Indicating with $\mathbf{a}$ the vector field which describes the evolution of the crack front, i.e. the local velocity of propagation assumed in the present case unknown in direction and modulus, the expression (10) becomes:

$$\overline{\boldsymbol{G}}_{V_\Gamma}(a_f) = \int_{S_\Gamma} \mathbf{a} \bullet (\sigma\,\mathrm{I} - \nabla \mathbf{u}^T \mathrm{T})\, \mathbf{n}\, dS \;-\; \int_{V_\Gamma} (\sigma\,\mathrm{I} - \nabla \mathbf{u}^T \mathrm{T}) \bullet \nabla \mathbf{a}\, dV \;. \quad (15)$$

Assuming $\mathbf{a}$ as the vector field which maximises $\overline{\boldsymbol{G}}_{V_\Gamma}(a_f)$, under the hypothesis of a quasi-static propagation phenomenon, the following equation can be written (Gurtin [3]):

$$\frac{d\overline{\boldsymbol{G}}_{V_\Gamma}(a_f)}{d\mathbf{a}} = \int_{S_\Gamma} (\nabla \mathbf{a}\, \mathbf{a}) \bullet (\sigma\,\mathrm{I} - \nabla \mathbf{u}^T \mathrm{T})\, \mathbf{n}\, dS \;-\; \int_{V_\Gamma} (\sigma\,\mathrm{I} - \nabla \mathbf{u}^T \mathrm{T}) \bullet \{\nabla(\nabla \mathbf{a})\}\, \mathbf{a}\, dV = 0 \,. \quad (16)$$

It can be demonstrated that there is a unique solution for the eqn (15). The eqn (16) can be seen as a linearized condition applied to the instantaneous shape of the crack front.

From eqn (15), when V_Γ has the *A* and *A'* bases parallel to the global $X_{\mathbf{1}} - X_{\mathbf{2}}$ plane:

$$\boldsymbol{G}(X_{\mathbf{3}}, a_f) = a_k \int_\Gamma (\sigma n_k - t_i u_{i,k})\, ds + a_k \int_A (\sigma \delta_{k3} - T_{i\mathbf{3}} u_{i,k})_{,\mathbf{3}}\, dS + \\ + a_{k,\mathbf{3}} \int_A (\sigma \delta_{k3} - T_{i\mathbf{3}} u_{i,k})\, dS - a_{k,\mathbf{3}} \int_A (\sigma \delta_{k3} - T_{i\mathbf{3}} u_{i,k})\, dS \quad . \qquad (17)$$

Therefore, in this case, eqn (14) assumes the general expression:

$$\boldsymbol{G}(X_{\mathbf{3}}, a_f) = \\ = a_{\mathbf{1}} \{ \int_\Gamma (\sigma n_1 - t_i u_{i,1})\, ds + \int_A (-T_{i\mathbf{3}} u_{i,1})_{,\mathbf{3}}\, dS \} + a_{\mathbf{3}} \{ \int_\Gamma (-t_i u_{i,3})\, ds + \int_A (\sigma - T_{i\mathbf{3}} u_{i,3})_{,\mathbf{3}}\, dS \} \qquad (18)$$

which can be written in the synthetic form:

$$\boldsymbol{G}(X_{\mathbf{3}}, a_f) = \boldsymbol{G_1}\, a_{\mathbf{1}} + \boldsymbol{G_3}\, a_{\mathbf{3}} \quad . \qquad (19)$$

From eqn (19) there is local fracture propagation if the following limit condition at a generic point along the crack front is verified:

$$\boldsymbol{G}_{\mathrm{Crit}}\, a = \boldsymbol{G_1}\, a_{\mathbf{1}} + \boldsymbol{G_3}\, a_{\mathbf{3}} \quad , \qquad (20)$$

where $\boldsymbol{G}_{\mathrm{Crit}}$ is a material property and the modulus $a = \sqrt{a_1^2 + a_3^2}$ is unknown.
With the same assumption used to obtain the eqn (11), when V_Γ has the *A* and *A'* bases parallel to the global $X_{\mathbf{1}} - X_{\mathbf{2}}$ plane, by means of tensor algebra calculations, the eqn. (16) becomes:

$$\int_\Gamma [\, a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} (\sigma n_1 - t_i u_{i,1}) + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} (\sigma n_3 - t_i u_{i,3})\,]\, ds + \\ + \int_{A+A'} [\, a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} (\sigma n_1 - t_i u_{i,1}) + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} (\sigma n_3 - t_i u_{i,3})\,]\, dS + \\ - \int_A [\, a_{\mathbf{3}} \frac{d^2 a_{\mathbf{1}}}{dX_3^2} (-T_{i\mathbf{3}} u_{i,1}) + a_{\mathbf{3}} \frac{d^2 a_{\mathbf{3}}}{dX_3^2} (\sigma - T_{i\mathbf{3}} u_{i,3})\,]\, dS = 0 \quad . \qquad (21)$$

Finally, transforming the surface integral on *A* + *A'*:

$$\int_\Gamma [\, a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} (\sigma n_1 - t_i u_{i,1}) + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} (\sigma n_3 - t_i u_{i,3})\,]\, ds + \\ - \int_A \{ \frac{d}{dX_3} [\, a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} (\sigma n_1 - t_i u_{i,1}) + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} (\sigma n_3 - t_i u_{i,3})\,] \}\, dS + \\ - \int_A [\, a_{\mathbf{3}} \frac{d^2 a_{\mathbf{1}}}{dX_3^2} (-T_{i\mathbf{3}} u_{i,1}) + a_{\mathbf{3}} \frac{d^2 a_{\mathbf{3}}}{dX_3^2} (\sigma - T_{i\mathbf{3}} u_{i,3})\,]\, dS = 0 \quad . \qquad (22)$$

The components of propagation velocity and their derivatives assume constant values on Γ and on $\boldsymbol{A}$, therefore ($n_1 = \mathbf{0}$, $n_3 = \mathbf{-1}$ and $t_i = -T_{ij} n_j = -T_{i\mathbf{3}}$ on $\boldsymbol{A}$):

$$
a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} \int_{\Gamma} (\sigma n_1 - t_i u_{i,1})\, ds + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} \int_{\Gamma} (-t_i u_{i,3})\, ds +
$$

$$
+ \frac{da_{\mathbf{3}}}{dX_3} \frac{da_{\mathbf{1}}}{dX_3} \int_{A} (-T_{i\mathbf{3}} u_{i,1})\, dS + a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} \int_{A} (-T_{i\mathbf{3}} u_{i,1})_{,\mathbf{3}}\, dS +
$$

$$
+ \left(\frac{da_{\mathbf{3}}}{dX_3} \right)^{\mathbf{2}} \int_{A} (\sigma - T_{i\mathbf{3}} u_{i,3})\, dS + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} \int_{A} (\sigma - T_{i\mathbf{3}} u_{i,1})_{,\mathbf{3}}\, dS = 0 \quad . \tag{23}
$$

$$
a_{\mathbf{3}} \frac{da_{\mathbf{1}}}{dX_3} G_{\mathbf{1}} + a_{\mathbf{3}} \frac{da_{\mathbf{3}}}{dX_3} G_{\mathbf{3}} + \frac{da_{\mathbf{3}}}{dX_3} \frac{da_{\mathbf{1}}}{dX_3} \int_{A} (-T_{i\mathbf{3}} u_{i,1})\, dS + \left(\frac{da_{\mathbf{3}}}{dX_3} \right)^{\mathbf{2}} \int_{A} (\sigma - T_{i\mathbf{3}} u_{i,3})\, dS = 0 \; . \tag{24}
$$

The eqn (24) and the eqn (20) (the two components of the velocity vector field are unknown) provide the shape's evolution of a quasi-statically propagating plane crack in a three-dimensional hyperelastic body.

4 CONCLUSIONS

Under classical hypotheses valid to apply the elastic fracture mechanics theory, a general expression of the potential energy release rate, $\overline{\boldsymbol{G}}$, for a three-dimensional problem has been set up. The general expression obtained in the paper takes into account the effects of a non uniform distribution of crack propagation rate along the fracture tip. Moreover imposing a linearized stationary condition to $\overline{\boldsymbol{G}}$ with respect to **a**, the crack propagation rate vector field, a system of two equations has been set up and it can be used to study analytically the evolution of a three-dimensional crack front shape. Suitable boundary conditions must be defined depending on the real geometry of the three-dimensional fractured body. The authors will execute future applications of the theory exposed in this paper to obtain practical results by means of dedicated finite element analyses and numerical solution of the crack evolution equations.

6 REFERENCES

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