RESONANCE AND CRACK PROPAGATION
IN PRESTRESSED ORTHOTROPIC MATERIALS CONTAINING AN INCLINED CRACK

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ABSTRACT

In a previous paper [6] we have established for a prestressed orthotropic material containing a crack the direction of crack propagation and the critical incremental stresses, using Sih’s generalised fracture criterion and considering all classical modes. Also, we concluded that in the case of prestressed isotropic materials acted by symmetrically distributed constant normal stresses the classical hypothesis used in Griffith-Irwin theory is justified according Sih’s new fracture criterion. For all three-fracture modes we observed that Griffith-Irwin theory and Sih’s new fracture criterion lead to the nearly same values of the critical incremental stresses producing crack propagation. In this paper we apply Sih’s generalised fracture criterion to determine the critical incremental stresses producing crack propagation as well as the direction of the crack propagation in a prestressed orthotropic elastic material containing an inclined crack. Also, using numerical analysis we study the resonance phenomenon for two prestressed composite materials.

KEYWORDS

Inclined crack, prestressed composite materials, Sih’s fracture criterion, resonance.

1. INTRODUCTION

We consider a prestressed material containing an inclined crack of a length \(2a\) situated in \(x_1x_3\) plane. We supposed that the material is unbounded and the crack faces are acted by constant normal incremental stresses \(\sigma\). The initial applied stress \(\sigma_0\) is in direction of the crack.

Our first aim is to determine the elastic state produced in the body using Guz’s representation theorem.

Our second aim is to determine the critical values of the incremental stresses and the direction of crack propagation. To do this, we use Sih’s generalised fracture criterion for an orthotropic and for transversally isotropic materials assuming that the body in not initially deformed. We assume that the stress free reference configuration of material is locally stable and initial deformation is infinitesimal.

In the last part we verify that the critical value of the initial applied stress \(\sigma_0\) for which the phenomenon of resonance can appear \(\sigma_0^*\) obtained by Guz [1] and Soós [2] is also available in the case of right crack parallel to the initial applied stresses.
2. GUZ’S REPRESENTATION THEOREM FOR INCREMENTAL FIELDS

We consider a prestressed, orthotropic, linear elastic material. We take as co-ordinate planes the symmetric planes of the material. We assume that the material is unbounded and contains a right crack of length 2a, situated in a plane making with \( x_2 \) \( x_3 \) an angle \( \beta \). We supposed that the material is prestressed by an initial applied stress \( \sigma_0 \) acting in the direction of \( x_1 \) - axis see Figure 1. We assume that the magnitude of the initial applied stress \( \sigma_0 \) is sufficiently small and it produces only infinitesimal initial deformations. We suppose that the upper and lower faces of the crack are acted by distributed incremental stress having a constant value \( p > 0 \), see Figure 1. Finally, we consider that the initial deformed equilibrium configuration of the body is locally stable.

![Image](image.png)

Figure 1: - Inclined crack in a prestressed orthotropic material

As it was shown by Guz [1,3,4] and independently by Soós [2] the incremental elastic state of the body can be expressed by two analytical complex potentials \( \Psi_j(z_j) \) defined in two complex planes \( z_j \), \( j=1,2 \). We denote by \( u_1, u_2 \) and respectively by \( \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} \) the involved components of incremental displacements, respectively incremental stresses.

According Guz’s representation formulae we have:

\[
\begin{align*}
\Phi_j &= 2 \text{Re} \left\{ b_j \Phi_1(z_1) + b_j \Phi_2(z_2) \right\} \quad ; \quad u_j = 2 \text{Re} \left\{ c_j \Phi_j(z_j) \right\} \\
\theta_{11} &= 2 \text{Re} \left\{ a_1 \mu_1^2 \Psi_1(z_1) + a_2 \mu_2^2 \Psi_2(z_2) \right\} \quad ; \quad \theta_{12} = -2 \text{Re} \left\{ \mu_1 \Psi_1(z_1) + \mu_2 \Psi_2(z_2) \right\} \\
\theta_{21} &= -2 \text{Re} \left\{ a_1 \mu_1 \Psi_1(z_1) + a_2 \mu_2 \Psi_2(z_2) \right\} \quad ; \quad \theta_{22} = 2 \text{Re} \left\{ \Psi_1(z_1) + \Psi_2(z_2) \right\}
\end{align*}
\]

(2.1)

(2.2)

(2.3)

In this relations

\[
\Psi_j(z_j) = \Phi_j(z_j) = \frac{d\Phi_j(z_j)}{dz_j}, \quad j=1,2
\]

(2.4)

and

\[
z_j = x_1 + \mu_j x_2, \quad j = 1,2.
\]

(2.5)

The parameters \( a_j, b_j, c_j, j=1,2 \) have following expressions

\[
a_j = (\sigma_{2112} \sigma_{1122} \mu_j^2 - \sigma_{1111} \sigma_{1212}) / (B_j \mu_j^2) ; \quad b_j = - (\sigma_{1112} + \sigma_{1212}) / B_j ; \quad c_j = (\sigma_{2112} \mu_j^2 + \sigma_{1111}) / (B_j \mu_j)
\]

(2.6)

(2.7)

The instantaneous elasticities \( \sigma_{klnm} \), \( k,l,m,n=1,2 \) can be expressed through the elastic coefficients \( C_{11}, C_{12}, C_{22} \) and \( C_{66} \) of the material and through the initial applied stress \( \sigma_0 \) by the following relations.

\[
\sigma_{1111} = C_{11} + \sigma_0 \sigma_{1212} = C_{66} ; \quad \sigma_{2222} = C_{22} \sigma_{2121} = C_{66} + \sigma_0 ; \quad \sigma_{1122} = C_{12} \sigma_{2112} = C_{66}.
\]

(2.8)

In their turn the elastic coefficients can be expressed using the engineering constants of the material and we have:

\[
C_{11} = \frac{1 - \nu_{22} \nu_{12}}{E_z E_y H} \quad ; \quad C_{22} = \frac{1 - \nu_{13} \nu_{23}}{E_1 E_2 H} \quad ; \quad C_{12} = \frac{\nu_{12} + \nu_{23} \nu_{13}}{E_1 E_2 H} \quad ; \quad C_{66} = G_{12}
\]

(2.9)

with

\[
H = (1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - \nu_{21} \nu_{32} \nu_{13} - \nu_{12} \nu_{23} \nu_{31}) / (E_1 E_2 E_3)
\]

(2.10)
In this relations $E_1$, $E_2$, $E_3$ are Young’s moduli in the corresponding symmetry directions of the material $\nu_{12}$, ..., $\nu_{32}$ are Poisson’s ratios and $G_{12}$, $G_{23}$, $G_{31}$ are the shear moduli in the corresponding symmetry planes. We recall that by $\sigma_0$ we have designed the initial applied stress acting in $x_1$ direction. The parameter $\mu_j$ can be obtained determining the roots $\nu_j$, $j=1,2$ of algebraic equation:

$$\nu^2 + 2A\nu + B = 0$$

with

$$A = \frac{[\sigma_{1111}\sigma_{2222} + \sigma_{1212}\sigma_{2112} - (\sigma_{1122} + \sigma_{1212})^2]}{\sigma_{2112}\sigma_{2222}}, \quad B = (\sigma_{1111}\sigma_{1221})/(\sigma_{2112}\sigma_{2222}).$$

As was shown by Guz [1, 3, 4] the above equation (2.11) can not have real roots if the initial deformed equilibrium configuration of the body is locally stable. The complex parameters $\mu_j$ can be calculated through the roots $\nu_j$, $j=1,2$ using equations (see Guz [1, 3, 4] and Soós [2])

$$\mu_1 = \sqrt{\nu_1}, \mu_2 = -\sqrt{\nu_2} \text{ if } \text{Im}\nu_j \neq 0, j=1,2$$

and

$$\mu_1 = \sqrt{\nu_1}, \mu_2 = -\sqrt{\nu_2} \text{ if } \text{Im}\nu_j = 0 \text{ and } \text{Re}\nu_j < 0, j=1,2.$$}

It can be shown that the parameters $\mu_j$, $j=1,2$ satisfy the relations

$$\text{Im}(\mu_1\mu_2) = 0 \quad \text{and} \quad \text{Re}(\mu_1 + \mu_2) = 0.$$

In what follows we assume that parameters $\mu_1$ and $\mu_2$ are not equal i.e.

$$\mu_1 \neq \mu_2.$$

The above condition is satisfied for orthotropic materials as well as for prestressed isotropic materials.

The expressions of the complex potentials $\Psi_j(z_j), j=1,2$ corresponding to our incremental boundary value problem were determined by Guz [1, 3, 4] and later independently by Soós [2] and have the following equations

$$\Psi_j(z_j) = \frac{(-1)^{j-1} a_j\mu_j K_j + K_H}{2\Delta\sqrt{2\sigma}}, \quad j=1,2$$

where

$$\Delta = a_2\mu_2 - a_1\mu_1$$

and

$$K_j = p\sin^2 \beta \sqrt{\pi a}, \quad K_H = p\sin \beta \cos \beta \sqrt{\pi a}$$

are the stress intensity factors corresponding to the first respectively second mode of fracture for an applied incremental stress $p>0$.

3. SIH’S ENERGETICAL CRITERION IN A MIXED FRACTURE MODE

We denote by $W$ the incremental strain energy density i.e.

$$W = \frac{1}{2} \theta_{ki} u_{i,k}, \quad k, l = 1,2$$

where

$$u_{i,k} = \frac{\partial u_i}{\partial x_k}$$

The expressions (2.17) of the complex potentials $\Psi_j(z_j), j=1,2$ show that near a crack tip the incremental strain energy density $W$ has a singular part as well a regular part. We design by $r$ and $\phi$ the radial distance from considered crack tip and the angle between radial direction and the line ahead the crack, see Figure 1.

Using the expressions (2.17) of complex potentials and Guz’s formulae (2.1) – (2.3) after long but elementary calculus we obtain that near, the considered crack tip the strain energy density have the following structure.
Here $S(\phi)$ is Sih's incremental strain energy density factor and is given by the following equation

$$S(\phi) = \frac{(a_1 \mu_1 K_1 + K_1)(a_2 \mu_2 K_1 + K_1)}{4\pi} s_m(\phi)$$  \hspace{1cm} (3.3)$$

where the function $s_m(\phi)$ depends on the elastic constants of the material and on the initial applied stress having the following expression:

$$s_m(\phi) = \text{Re} \left[ \frac{a_1 a_2 \mu_1 \mu_2}{\Delta} \left( \frac{\mu_1}{\chi_1(\phi)} - \frac{\mu_2}{\chi_2(\phi)} \right) \right] \text{Re} \left[ \frac{1}{\Delta} \left( \frac{a_1 a_2 b_1}{\chi_1(\phi)} - \frac{a_1 \mu_1 b_2}{\chi_2(\phi)} \right) \right] + \ldots$$  \hspace{1cm} (3.4)$$

In above equation

$$\chi_j(\phi) = \sqrt{\cos \phi + \mu_j \sin \phi}, j = 1, 2.$$  \hspace{1cm} (3.5)$$

We extend the validity of Sih's fracture criterion, (see [5]) for orthotropic or isotropic prestressed elastic materials assuming that:

H1: Crack propagation will start in a radial direction $\phi_c$ along with the incremental strain energy density factor $S(\phi)$ is a minimum, i.e.

$$\frac{dS}{d\phi}(\phi_c) = 0, \quad \frac{d^2S}{d\phi^2}(\phi_c) > 0.$$  \hspace{1cm} (3.6)$$

H2: The critical intensity

$$S_c = S_{\text{min}} = S(\phi_c)$$  \hspace{1cm} (3.7)$$

governs the onset of the crack propagation and it represents a material constant independent of the crack geometry, loading and initial applied stress. The assumed Sih’s type fracture criterion is based on local density of the incremental strain energy near the crack tip and requires no apriori assumptions concerning the direction in which energy is released by separating crack surfaces. Using the equations (2.19), (3.3) and (3.7) we obtain the incremental stress $p_c$ for which crack propagation starts at critical direction $\phi_c$

$$ap_c^2 = 4S_c/s_m(\phi_c).$$  \hspace{1cm} (3.8)$$

In above relation $S_c$ is Sih’s new material parameter, which takes the place of Griffith’s specific surface energy $\gamma$ in the new theory of brittle fracture. Once $S_c$ is known, the relation (3.8) can be used to get $p_c$.

4. CRACK PROPAGATION FOR UNPRESTRESSED ORTHOTROPIC MATERIALS

In this section we suppose that our material is not initially deformed i.e.:

$$\sigma_0 = 0.$$  \hspace{1cm} (4.1)$$

Long, but elementary calculus shows that in this case we have

$$a_1 = a_2 = 1.$$  \hspace{1cm} (4.2)$$

Hence, according to equation (2.18) we get

$$\Delta = \mu_2 - \mu_1.$$  \hspace{1cm} (4.3)$$

Obviously, the relation (3.3) giving Sih’s strain energy factor $S(\phi)$ rest valuable, but now the function $s_m(\phi)$ will have a simplified form. In the following we present the results of our numerical analysis concerning the possible values of critical angle $\phi_c$ versus angle $\beta$ for two anisotropic materials. For simplicity we have assumed transversally isotropic materials, $x_2 x_3$ being isotropy plane. In this case we have

$$E_2 = E_1, \nu_{12} = \nu_{13}, \nu_{23} = \nu_{32}, G_{12} = G_{31}, \nu_{21} = \nu_{31}$$  \hspace{1cm} (4.4)$$

and

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad G_{23} = \frac{E_2}{2(1 + \nu_{23})}.$$  \hspace{1cm} (4.5)$$
We consider two transversally isotropic materials characterised by the following values

(i) \( E_2 = 5.56 \text{GPa}, \nu_{23} = 0.2, G_{23} = 2.3 \text{GPa}, E_1 = 75 \text{GPa}, \nu_{12} = 0.22, G_{12} = 2 \text{GPa} \) \hspace{1cm} (4.6)

and

(ii) \( E_2 = 10.3 \text{GPa}, \nu_{23} = 0.28, G_{23} = 4.02 \text{GPa}, E_1 = 181 \text{GPa}, \nu_{12} = 0.46, \ G_{12} = 2 \text{GPa} \) \hspace{1cm} (4.7)

The set (i) corresponds to a fibre reinforced aramid / epoxy composite and the set (ii) corresponds to a fibre reinforced graphite / epoxy composite. The dependence of the crack propagation angle \( \varphi_c \) versus angle \( \beta \) for the two materials (i) and (ii) is presented in Figure 2. We observe that critical angle \( \varphi_c \) decrease when crack’s angle \( \beta \) increases. Also, a remarkable result is obtained in the case when, \( \beta = \frac{\pi}{2} \) which corresponds to the first classical fracture mode. In this case we obtain a well-known result that the crack will propagate along its line (see [6]).

5. RESONANCE PHENOMENON FOR AN INCLINED CRACK

As Guz [3] shows the phenomenon of internal stability or resonance has a well-defined physical meaning and can be elucidated taking into account the fact that all materials have an internal structure. The phenomenon concerns the loss of stability of the structure and depends on the geometrical and mechanical characteristics of the body as a whole. A rigorous study of this phenomenon has to take into account explicitly the parameters describing the internal structure of the body. Taking into account that such phenomenon can take place arises the following question: May exist a critical value \( \sigma_0^c \) of the initial applied stress \( \sigma_0 \) such that when \( \sigma_0 \) starting from zero converges to \( \sigma_0^c \), the incremental stress \( p_c \) converges to zero? For a fibre reinforced composite the answer to this question was given by Guz and it is positive (see [1], Chap. 2). How for a fibre reinforced composite material Young’s modulus \( E_1 \) is greater than \( E_2 \) and than shear modulus \( G_{12} \), Guz was able to show that the critical value \( \sigma_0^c \) is given by the following relation:

\[
\sigma_0^c \approx -G_{12} \left\{ 1 - \frac{G_{12}^2}{E_1 E_2} (1 - \nu_{12} \nu_{31}) (1 - \nu_{23} \nu_{31}) \right\} < 0. \tag{5.1}
\]

Since \( E_2 \ll E_1 \) and \( G_{12} \ll E_1 \) the critical compression stress \( \sigma_0^c \) produces only infinitesimal strains in the prestressed material. In what follows we verify in the case \( \beta \) is constant, for our reinforced materials (i) and (ii) characterised by the values given by eqs (4.6) respectively (4.7) that phenomenon of resonance can appear when \( \sigma_0^c \) converges to \( \sigma_0_{(i)}^c \) for material (i) and to \( \sigma_0_{(ii)}^c \) for material (ii). Here we denoted by \( \sigma_0_{(i)}^c \) and respectively \( \sigma_0_{(ii)}^c \), the critical value for materials (i) and respectively (ii) from expression (5.1) and we obtained the following values:
\[ \sigma_{0(i)}^c = -1.9805 \text{GPa} \quad \text{and} \quad \sigma_{0(ii)}^c = -1.9957 \text{GPa}. \]  
Using equation (3.8) we obtain for the critical incremental stress following expression:
\[ p_c = 2 \sqrt{\frac{S_c}{a S_m(\phi_c)}}. \]
Table 1 presents the dependence of the incremental stress \( \phi_c \) and of critical fracture angle \( \phi_c \) for materials (i) and (ii). How Sih’s parameter \( S_c \) and the fracture’s length are constant we shall study the variation of the new stress function \( \pi_c = \pi_c(\sigma_0) \) given by:
\[ \pi_c = \frac{p_c}{2} \sqrt{\frac{a}{S_c}}. \]

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<th>Material</th>
<th>( \sigma_0 )</th>
<th>( \pi_c ) (GPa)</th>
<th>( \phi_c ) (°)</th>
<th>( \pi_c ) (GPa)</th>
<th>( \phi_c ) (°)</th>
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6. FINAL REMARKS

From our numerical analysis we conclude that: 
- critical fracture angle \( \phi_c \) decreases when crack’s angle \( \beta \) increases; 
- when crack’s angle \( \beta \) is equal with \( \pi/2 \) case corresponding to the first classical fracture mode we obtain a well-known result, that \( \phi_c = 0 \), i.e. the crack propagates along its line; 
- when \( \sigma_0 \) decreases to the critical value \( \sigma_0^c \) also the incremental stress \( p_c \) decrease. When \( \sigma_0 \) converges to \( \sigma_0^c \) we observe that \( p_c \) converges to zero, i.e. the crack will start to propagate even that \( p_c \) is approximately zero. In this case resonance phenomenon occurs. 
- critical fracture angle \( \phi_c \) decreases when incremental stress \( \sigma_0 \) decreases. In the case \( \beta = \pi/2 \), which corresponds to the first classical fracture mode, for \( \sigma_0 = 0 \) we obtain that critical fracture angle \( \phi_c \) is in a vicinity of 40°, (see [6]).

REFERENCES