NUMERICAL REPRESENTATION OF PLASTIC *J*-INTEGRAL VARIATION ALONG THE CRACK FRONT OF SEMI-ELLIPTICAL SURFACE CRACK UNDER UNIFORM TENSION.

Monir H. Sharobeam

Richard Stockton College of New Jersey,
Pomona, NJ 08240 USA

ABSTRACT

A mathematical form that represents the average plastic *J*-integral over the crack front of a semi-elliptical surface crack under uniform tension has been recently developed by the author. The form provides an estimate for the average plastic *J-Integral* at any stage of loading for wide array of surface crack geometries and material's hardening exponent. The purpose of this study is to introduce a numerical representation of the variation of the local plastic *J*-integral along the crack front. This is developed by studying the relationship between the average plastic *J*-integral and the local plastic *J*-integral at any point of the crack front. This relationship depends on the parametric angle, crack depth, crack width, and probably also geometry thickness to width ratio and the material hardening exponent. A wide array of crack geometries are used in this study. The crack depth to geometry thickness ratio varied from a shallow crack with a ratio of 0.1 to a deep crack with a ratio of 0.82. The crack depth to crack width ratio ranged from a narrow crack with a ratio of 2.0 to a wide crack with a ratio of 0.5. The finite element method has been used to develop the different models. About 35 different cases are included in the study to encompass the wide variation in the studied parameters.

KEYWORDS

elastic-plastic fracture; J-integral; surface cracks; load separation.

INTRODUCTION

The energy rate interpretation of *J*-integral, which was introduced by Rice [1] for two-dimensionsl geometries, can be applied for both: elastic *J*-integral, J_{el}, and plastic *J*-integral, J_{pl}. Hence, J_{pl} for two-dimensional geometries can be written as [1,2]:

$$J_{pl} = -\frac{1}{B}\left[\frac{\partial U_{pl}}{\partial a}\right]_{v_{pl}} \tag{1}$$

where U_{pl} is the plastic potential energy, B is specimen width, a is crack length and v_{pl} is the plastic load line displacement. It was also domenstrated that the relation between the load per unit thickness, P, and v_{pl} in two-dimensional geometries can be represented by a separable form [2,3,4] as:

$$P = G(a)H(v_{pl}) \tag{2}$$

where $G(a)$ and $H(v_{pl})$ are the geometry and deformation functions. The energy rate interpretation of J_{pl}, as represented by Eq. (1), and the load separation as represented by Eq. (2) led to the development of the single specimen J_{pl} expression for these geometries which can be written as:

$$J_{pl} = \frac{\eta_{pl}}{b} A_{pl} \tag{3}$$

where b is the uncracked ligament, A_{pl} is the area under the P versus v_{pl} record and η_{pl} is a factor that depends on the geometry and crack length [5,6,7].

The energy rate interpretation form given in Eq. (1) can be also written as:

$$-\left[\Delta U_{pl}\right]_{v_{pl}} = B\, J_{pl}\; \Delta a \tag{4}$$

This expression indicates that the infinitesmal difference in the plastic potential energy between two identical specimens with an arbitrary infinitesmal crack length difference of Δa at the same v_{pl} is equal to the crack driving force, $[BJ_{pl}]$, times the crack length difference. In three-dimensional geometries, such as surface cracks, this expression can be written as [8]:

$$-\left[\Delta U_{pl}\right]_{v_{pl}} = \int_S J_{pl}\; (\Delta a)\; dS \tag{5}$$

where S represents the crack front of the surface crack. J_{pl} varies along the crack front while Δa is arbitrary according to equation (4) and can be assigned a position function or a specific effective value along the crack front [8]. This can lead to an energy rate interpretation form for J_{pl} for surface cracks as:

$$J_{pl,av} = -\frac{1}{S}\left[\frac{dU_{pl}}{da}\right]_{v_{pl}} \tag{6}$$

where $J_{pl,av}$ is the average plastic J-integral across the crack front and can be written as:

$$J_{pl,av} = \frac{1}{S}\int_S J_{pl}\; dS \tag{7}$$

Sharobeam and Landes demonstrated load separation in the test records of surface crack geometry under uniform tension [8,9,10] for plastic load line displacement and also plastic crack mouth opening displacement. A separable load expression for this geometry can be written as:

$$\frac{\sigma_t}{\sigma_o} = G\left(\frac{a}{t}, \frac{a}{c}, \frac{t}{W}\right) H\left(\frac{v_{pl}}{t}\right) \tag{8}$$

where σ_t is the remote tensile stress, σ_o is yield stress, a is crack depth and c is crack width as shown in Fig. 1. t and W are specimen thickness and width, respectively. Load separation and the energy rate interpretation of $J_{pl,av}$ lead to a single specimen form for $J_{pl,av}$ for this geometry that can be written as:

$$J_{pl,av} = \zeta_{pl} \int \sigma_t\; dv_{pl} \tag{9}$$

The factor ζ_{pl} here is equivalent to (η_{pl}/b) in the single specimen J_{pl} form for two-dimensional geometries, as represented by Eq. (3), and it is a function of the crack depth to width ratio, a/c, crack depth to thickness ratio, a/t, and specimen thickness to width ratio, t/W. Sharobeam and Landes [8] demonstrated also that the ratio between plastic load line displacement and plastic crack mouth opening displacement is independent of the amount of deformation which allows using either in Eqs. (8) and (9) as v_{pl}. They also developed a mathematical representation for the single specimen $J_{pl,av}$ form using the plastic crack mouth opening displacement as:

$$J_{pl,ave} = \frac{n}{n+1}(\sigma_o t)\zeta_{pl}\ (G)\left(\frac{v_{pl}}{t}\right)^{\frac{n}{n+1}} \tag{10}$$

where n is the material hardening exponent and G is the geometry function. This expression is developed for Ramberg-Osgood materials, which follow the stress-strain relationship given as:

$$\frac{\varepsilon}{\varepsilon_o} = \frac{\sigma}{\sigma_o} + \alpha\left(\frac{\sigma}{\sigma_o}\right)^n \tag{11}$$

where ε is strain, ε_o is yield strain and α is a constant. Detailed expressions for the ζ_{pl} factor and the geometry function G are given in references [8] and [11], respectively, for surface cracks with wide range of a/t, a/c, t/W and n values.

The purpose of this study is to examine the variation of local J_{pl} along the crack front of semi-elliptical surface cracks and develop a relationship between the ratio of local J_{pl} to $J_{pl,av}$ and the location at the crack front represented by the parametric angle θ as shown in Fig. 2. Using this relationship together with Eq. (10), local J_{pl} at any point of the crack front of a surface crack can be obtained. To study J_{pl} variation along the crack front of a surface crack and develop the required relationship, numerical records of J-integral versus θ for wide array of semi-elliptical surface cracks were generated using the finite element method. Material was selected to be a Ramberg-Osgood material with a hardening exponent, n, equal to 5. The specimen width and half height were selected to be eight times the specimen thickness. Crack depth to thickness ratio varied from 0.1 to 0.82 while crack depth to crack width ratio varied from 0.5 to 2.0.

THE FINITE ELEMENT MODEL

The symmetry of the geometry allowed the consideration of only a single quarter of the geometry in the model. The model contains 3042 nodes and 402 20-node hybrid brick elements. The crack vicinity is represented by six rings of focused elements as shown in Fig. 3. Each ring contains 36 elements; six elements along the crack front times six along half the circumference of the ring. The structure of this model is close to those used by Kirk and Dodds [12] and Kim and Hwang [13]. To capture the plastic and elastic singularities at the crack tip, the crack tip side of the elements in the first ring are collapsed, the mid-side nodes of these elements are moved to the quarter point and the size of the elements in the first few rings are increased proportionally to the square root of their distance from the crack tip. A FORTRAN program was developed to generate the mesh for the model according to the selected a/t, a/c and t/W values. The program also generated the input file for the finite element code used in this study, ABAQUS. In an early study [8], results of this model have been found to be in a good agreement with the experimental elastic-plastic test records of specimens with identical crack geometries [11] and also Newman-Raju numerical solutions for elastic J-integral. The results of this model were also found in good agreement with the results of a more refined model with similar structure but additional 2555 nodes and 396 element just in the crack vicinity [11].

FINITE ELEMENT RESULTS

To study the effect of a/t on the distribution of $J_{pl}/J_{pl,av}$ along the crack front for different a/c values, models with a/c=0.5, 0.75, 1.0, 1.5 and 2.0 and a/t that varies from 0.1 to 0.82 were developed. For each crack geometry, the model was run twice; one with a linearly elastic material and another with a Ramberg-Osgood elastic-plastic material. The plastic component of J was obtained by subtracting J_{el} obtained using the elastic run from the total J obtained using the elastic-plastic run for the same load. $J_{pl,av}$ was, then, evaluated by integrating J_{pl} over the crack front as indicated by Eq. (7). The $J_{pl}/J_{pl,av}$ distribution along the crack front for the different crack geometries are shown in Figs. 4-8. It is clear from these figures that there are two trends for $J_{pl}/J_{pl,av}$ distribution along the crack front. For shallow, wide cracks ($a/t \leq 0.4$ and $a/c \leq 0.75$), the distribution begins with a low value at the surface ($\theta = 0$), then increases until it reaches maximum at the deepest point of the crack

($\theta = 90°$). For deep, wide cracks ($a/t \geq 0.6$ and $a/c \leq 0.75$) and all circular ($a/c = 1.0$) and narrow cracks ($a/c > 1.0$), the $J_{pl}/J_{pl,av}$ distribution begins also with a low value at the surface, reaches a maximum value mostly between $\theta = 15°$ and 30° then decreases gradually as θ increases. The maximum $J_{pl}/J_{pl,av}$ values for surface cracks that represent a transition between the two trends such as the two surface cracks: (a/t = 0.6 , a/c = 0.5) and (a/t = 0.5, a/c = 0.75) occurred between $\theta = 45°$ and 60°. For the cases where $J_{pl}/J_{pl,av}$ reaches a maximum before decreasing, it can be noticed that the rate at which $J_{pl}/J_{pl,av}$ decreases as θ increases depends on a/t such that the deeper the crack the higher the rate. Similar trends for normalized J distribution were obtained by Yagawa, Ueda and Takahashi [15] for four different surface cracks with $a/t = 0.4$ and 0.8 and $a/c = 0.2$ and 0.6. Kirk and Dodds [12] results also for a surface crack with $a/t = 0.13$ and $a/c = 0.38$ showed similar trend to that of the shallow, wide surface cracks in this study. The low value for $J_{pl}/J_{pl,av}$ at the surface is contributed to the low constraint condition which results in a reduced order of singularity in the surface boundary layer[12]. From the figures, it is also clear that, for each of the studied a/c values, the $J_{pl}/J_{pl,av}$ values for shallow cracks (a/t = 0.1 to 0.4) are very close for the whole crack front. The difference is less than 6% at any point on the crack front. On the other hand, there is a wide scatter for the $J_{pl}/J_{pl,av}$ values for medium to deep cracks ($a/t > 0.4$ to 0.8) with $a/c = 0.5$, 0.75 and 1.0. This scatter, however, becomes narrow as a/c reaches 1.5 and collapses into almost a single line for the different surface cracks with $a/c = 2.0$.

To make sure that these distributions are uniquely related to the crack geometry (a/t and a/c) and not dependent on the amount of loading, $J_{pl}/J_{pl,av}$ was evaluated for four different surface cracks at different levels of loading. The results of these evaluations are given in Table 1 which shows that the $J_{pl}/J_{pl,av}$ values are almost identical for the different loading levels over the whole crack front except for the point at the surface ($\theta = 0$). This indicates that $J_{pl}/J_{pl,av}$ distribution along the crack front of a surface crack is independent of the loading level at least for the loading range included in this study which is from a little below the yield value up to almost twice the yield value.

NUMERICAL REPRESENTATION OF THE RESULTS

Several functions have been tried to provide a general fit for $J_{pl}/J_{pl,av}$ distribution along the crack front. It was difficult to find a function that can capture the two different trends and provide a perfect fit for all the data points along the crack front. The following function simulated the two different trends of distribution with a reasonably close fit.

$$f_1(\theta) = A_1 \sin\theta \quad + B_1\sqrt{\sin\theta} \quad + C_1\sqrt[3]{\sin\theta} \quad + D_1 \tag{12}$$

where $f_1(\theta)$ represents $J_{pl}/J_{pl,av}$ and A_1, B_1, C_1 and D_1 are the fit function coefficients. Figs. 9-12 show how this function fits the finite element data for $J_{pl}/J_{pl,av}$ versus θ distribution for four different cracks. This fit function is labelled as the first fit function in these figures. It provides an excellent fit for all data points except for a narrow range around the maximum point for the cases where the maximum occurs at low θ values ($\theta < 45°$). At these maximum points, the fit is off the finite element data by less than 4%.

When the surface point ($\theta = 0$), where $J_{pl}/J_{pl,av}$ value showed a little dependence on the load level, is not include in the fit, a simple polynomial function of the third order provided also a reasonable fit for both trends. It was off by less than 2% for most of the data points except again where there is a maximum at θ below 45°. The difference between the fit and the finite element value at the maximum point is typically less than 4%. This fit function is shown in Figs. 9-12 and is labelled as the second fit function. It can be written as:

$$f(\theta) = A\sin^3\theta \quad + B\sin^2\theta \quad + C\sin\theta \quad + D \tag{13}$$

where $f(\theta)$ represents $J_{pl}/J_{pl,av}$ and A, B, C and D are the fit function coefficients. This function is applicable for the range $\theta = 7.5°$ to 90° where $J_{pl}/J_{pl,av}$ values are found to be independent of the loading level. Because of the simplicity of this function and it is reasonable representation of the $J_{pl}/J_{pl,av}$ distribution along the crack front, it is adopted in this study. Table 2 lists the coefficients A, B, C and D for wide array of surface cracks.

TABLE 1: $J_{PL}/J_{PL,AV}$ VERSUS θ FOR DIFFERENT SURFACE CRACKS AT DIFFERENT LOADING LEVELS.

Crack	*a/t*=0.6, *a/c*=1.63			*a/t*=0.3, *a/c*=0.75			*a/t*=0.4, *a/c*=2.0			*a/t*=0.82, *a/c*=1.0		
	σ_t/σ_o =			σ_t/σ_o =			σ_t/σ_o =			σ_t/σ_o =		
θ	0.9	1.1	1.3	1.2	1.5	1.8	1.1	1.3	1.5	0.9	1.0	1.1
0.0	0.69	0.78	0.83	0.43	0.47	0.48	0.86	0.91	0.93	0.65	0.70	0.73
7.5	1.17	1.17	1.16	0.79	0.78	0.77	1.24	1.24	1.24	1.30	1.29	1.28
15.0	1.32	1.31	1.30	0.89	0.88	0.87	1.35	1.35	1.35	1.44	1.42	1.41
30.0	1.18	1.18	1.18	0.99	0.98	0.98	1.17	1.17	1.17	1.22	1.22	1.22
45.0	0.99	0.99	1.00	1.04	1.04	1.04	0.96	0.96	0.96	0.98	0.98	0.98
60.0	0.85	0.84	0.84	1.07	1.08	1.08	0.78	0.78	0.78	0.86	0.86	0.87
75.0	0.74	0.73	0.73	1.10	1.10	1.10	0.66	0.65	0.65	0.75	0.75	0.75
90.0	0.71	0.70	0.69	1.11	1.11	1.12	0.62	0.61	0.60	0.70	0.70	0.70

Using this fit function, $J_{pl}/J_{pl,av}$ distribution along the crack front of surface cracks with *a/t*=0.1, 0.2,…or 0.8 and *a/c*=0.5, 0.75, 1.0, 1.5 or 2.0 can be reconstructed. For surface cracks with different *a/t* and *a/c* values but within the studied ranges, $J_{pl}/J_{pl,av}$ distribution can be obtained using one or two-dimensional fitting functions between the $J_{pl}/J_{pl,av}$ distributions of neighboring surface cracks. Because of the small increments in *a/t*, a linear or quadratic fit function can be used to construct $J_{pl}/J_{pl,av}$ distribution for a surface crack geometry with *a/t* that is not included in the given array of surface cracks. For surface cracks with *a/c* values that are not included in the array, a second or a third order polynomial fit is recommended. There are many commercial math packages that provide one and two-dimensional fitting functions. Using $J_{pl}/J_{pl,av}$ distributions constructed by the fit functions given in Table 1 for surface cracks with same *a/t* but different *a/c* values in a third order polynomial surface fit, $J_{pl}/J_{pl,av}$ distributions a surface crack geometry with a new *a/c* value that is within the range of the included *a/c* values can be obtained. Figs. 13 and 14 show a comparison between $J_{pl}/J_{pl,av}$ distributions obtained by the finite element model and Mathcad 2000 two-dimensional fit for two surface cracks geometries with new *a/c* values. The two-dimensional fit provided $J_{pl}/J_{pl,av}$ distribution that is very close to the finite element data.

TABLE 2: THE FIT FUNCTION COEFFICIENTS FOR A WIDE ARRAY OF SURFACE CRACKS.

		a/t=							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
a/c=0.5	*A*=	-0.030	0.152	0.165	0.099	0.096	0.310	0.427	0.535
	B=	0.118	-0.193	-0.322	-0.365	-0.695	-1.381	-1.926	-2.462
	C=	0.592	0.826	0.943	1.003	1.185	1.602	1.845	2.059
	D=	0.546	0.458	0.438	0.453	0.526	0.519	0.605	0.711
a/c=0.75	*A*=	0.226	0.482	0.597	0.611	0.530	0.930	1.491	0.997
	B=	-0.569	-1.036	-1.309	-1.471	-1.517	-2.454	-3.606	-2.690
	C=	0.799	1.043	1.187	1.285	1.341	1.809	2.258	1.472
	D=	0.677	0.644	0.641	0.657	0.682	0.686	0.746	1.014
a/c=1.0	*A*=	0.628	0.827	0.912	0.664	0.839	1.295	1.748	1.909
	B=	-1.483	-1.822	-2.025	-1.639	-2.114	-3.096	-3.953	-4.040
	C=	1.022	1.203	1.323	1.123	1.397	1.879	2.149	1.781
	D=	0.834	0.804	0.793	0.834	0.822	0.811	0.875	1.092
a/c=1.5	*A*=	1.159	1.243	1.328	1.407	1.463	1.397	1.590	1.871
	B=	-2.740	-2.894	-3.050	-3.238	-3.381	-3.360	-3.610	-3.958
	C=	1.304	1.411	1.520	1.635	1.706	1.713	1.671	1.623
	D=	1.036	1.009	0.983	0.969	0.972	0.984	1.053	1.137
a/c=2.0	*A*=	0.985	1.103	1.181	1.311	1.323	1.452	1.502	1.314
	B=	-2.478	-2.720	-2.882	-3.150	-3.222	-3.491	-3.597	-2.994
	C=	1.043	1.142	1.204	1.311	1.372	1.523	1.550	1.019
	D=	1.129	1.131	1.136	1.140	1.132	1.115	1.129	1.253

CONCLUSION

$J_{pl}/J_{pl,av}$ distribution along the front of surface crack geometries with wide ranges of *a/t* and *a/c* values were developed using the finite element method. Width and height of each geometry were selected to be eight times the thickness and the material is considered as a Ramberg-Osgood material with a hardening exponent of 5. Two trends for $J_{pl}/J_{pl,av}$ distributions along the crack front have been observed. For shallow, wide cracks, the $J_{pl}/J_{pl,av}$ ratio begins with a low value at the surface then increases until it reaches its maximum at the deepest point. For most of other studied crack geometries, the $J_{pl}/J_{pl,av}$ ratio begins with a low value at the surface but reaches a maximum at θ below 45°, then decreases as θ increases. It was also observed that $J_{pl}/J_{pl,av}$ distribution for narrow surface cracks (*a/c* = 1.5 and 2) is less dependent on *a/t* than that for circular (*a/c* = 1.0) and wide (*a/c* = 0.5 and 0.75) cracks. A general fit function that encompasses both trends was developed. Such a function can be used to construct $J_{pl}/J_{pl,av}$ distribution for any surface crack with *a/t* and *a/c* within the studied ranges. Using the constructed $J_{pl}/J_{pl,av}$ distribution and the mathematical representation of $J_{pl,av}$ given in Eq. (10), a full representation of J_{pl} along the crack front at any stage of loading can be predicted. Other parameters that may influence $J_{pl}/J_{pl,av}$ distribution such as thickness to width ratio and material hardening exponent are now under consideration by the author.

REFERENCES

1. Rice, J. R. (1968) *Journal of Applied Mechanics,* 35, 379-386.
2. Paris, P. C., Ernst, H. and Turner, C. E. (1980) *Fracture Mechanics - Twelfth Conference,* ASTM STP 700, 338-351.
3. Sharobeam, M. H., and Landes, J. D. (1991) *International Journal of Fracture Mechanics*, 47, 81-104.
4. Sharobeam, M. H., and Landes, J. D. (1993) *International Journal of Fracture Mechanics*, 59, 213-226.
5. Rice, J. R., Paris, P. C., and Merkle, J. G. (1973) *Progress in Flaw Growth and Fracture Toughness Testing, ASTM STP 536*, 231-245.
6. Merkle, J. G. and Corten, H. T. (1974) *PVP-33, ASME*, 286-292.
7. Clarke, G. A. and Landes, J. D. (1979) *Journal of Testing and Evaluation, JTEV*, A7, 5,264-265.
8. Sharobeam, M. H., and Landes, J. D. (1994) *Fracture Mechanics: 25th Symposium, ASTM STP 1220*, 397-414.
9. Sharobeam, M. H. (1994) *FAA/NASA International Symposium on Advanced Structural Integrity Methods for Airframe Durability and Damage Tolerance, NASA CP-3274*, 2, 703-724.
10. Sharobeam, M. H., and Landes, J. D. (1993) *International Journal of Fracture*, 61, 379-395.
11. Sharobeam, M. H. and Landes, J. D. (1993) *Engineering Fracture Mechanics,* 63, 131-145.
12. Kirk, M. T. and Dodds, R. H., Jr. (1991) *Engineering Fracture Mechanics*, 39, 3, 535-551.
13. Kim, J. H. and Hwang, I. S. (1998) *PVP-365, ASME,* 325-331.
14. Newman, J. C. and Raju, I. S. (1983) *Fracture Mechanics: 14th Symposium, ASTM STP 791*, 283-265.
15. Yagawa, G., Ueda, H. and Takhashi, Y. (1986) *PVP-103, ASME,* 43-48.