INTRODUCTION OF THE HAMILTONIAN SYSTEM INTO THE ASYMPTOTIC ANALYSIS IN FRACTURE MECHANICS

J. LI

Laboratoire d'Etudes et de Recherches en Mécanique des Structures
Blaise Pascal University of Clermont-Ferrand
Avenue Aristide Briand, 03100 Montluçon, France

ABSTRACT

The asymptotic analysis is a powerful tool in fracture mechanics. The principal criteria of the fracture toughness, such like *K* and *J*, were established on the basis of the asymptotic analysis of the crack-tip fields. In traditional theories, asymptotic analyses are often carried out by using the Airy stress function in establishing high-order differential equations governing the near-tip fields. However, as far as the problems posed in fracture mechanics become complicate, the establishment and the solution of the high-order differential equations become more and more difficult. In this paper, we make use of another philosophy in the asymptotic analysis in fracture mechanics. The key idea is to introduce the Hamiltonian system, usually studied in rational mechanics, into continuum mechanics. We present in this paper the guideline of this method through a simple bi-dimensional elastic problem. The establishment of the governing equations was performed by imitating either the radial coordinate or the angular coordinate as "time". By defining the dual variables in the state space, systems of the first-order differential equations were obtained. The solution methods are also discussed in the frame of eigenvalue computations. It was shown that this method is very efficient and quite simple in establishment and solution of the governing equations of this class of problems. It allows considerations of a large range of problems which are difficult for traditional methods.

KEY WORDS

Near-tip fields, asymptotic analysis, Hamiltonian system, eigenvalue problems

INTRODUCTION

The asymptotic analysis is a powerful tool in fracture mechanics. It permits to write the near-tip fields of a crack or a notch, singular or not, under expansion form. The intensity of these fields can be determined by considering the boundary conditions of the structure by means of some well-chosen parameters. In general, the first term in the expansion suffices to characterize the near-tip fields. If it is not accurate enough, other terms can be added. The principal criteria of the fracture toughness, such like *K* and *J*, were founded on the basis of the asymptotic analysis of the crack-tip fields.

Williams (1957) first carried out asymptotic study near a crack-tip in bi-dimensional elastic material. Airy stress function was used in establishing a fourth order differential equation governing the near-tip fields. Afterward, this method was essentially followed in the asymptotic analysis in fracture mechanics. One can list numerous important results found by using this method. However, as far as the problems posed in the fracture mechanics become complex, the establishment and the solution of the high-order differential

equation become more and more difficult. In certain cases, the stress function may not exist; this makes the traditional method inadequate in these studies.

In this paper, we will make use of another method in the asymptotic analysis in fracture mechanics. The key idea of this new method is to introduce the Hamilton principle, usually used in rational mechanics, into continuum mechanics. Zhong (1995) systematically described this methodology in solving elasticity problems. We believe that it will be helpful to introduce it into fracture mechanics, especially in the asymptotic analysis of the near-tip fields.

Rational mechanics often makes use of the Lagrange function (cinematic energy – potential energy) to describe a dynamic system. It can be represented as function of the generalized displacement vector and its deviation with respect to time, i.e. $L(\boldsymbol{q},\dot{\boldsymbol{q}})$. The Hamilton principle is described as follows: the true movement trajectory from time t_0 to time t_1 of a conservative system makes the action A minimal:

$$\delta A = \delta \int_{t_0}^{t_1} L(\boldsymbol{q},\dot{\boldsymbol{q}})dt = 0 \qquad (1)$$

This variational principle leads to the Langrage equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \qquad (2)$$

By introducing the vector of generalized momentum $\boldsymbol{p} = \dfrac{\partial L}{\partial \dot{\boldsymbol{q}}}$ and the Hamilton function $H(\boldsymbol{q},\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{q} - L(\dot{\boldsymbol{q}},\boldsymbol{q})$, the Lagrange equation can equivalently be written under form of the Hamilton canonical equations:

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} \quad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} \qquad (3)$$

Return back now to fracture mechanics. We can actually write the equations governing the crack tip fields under form of (3). The main idea is to consider one coordinate in the polar system as "time" and take the total potential energy as the Lagrange function. For example, we can consider the radial coordinate r or the angular coordinate θ as time and take the variational principles established in continuum mechanics as the Hamilton variational principle. Then all the procedures currently used in rational mechanics can be translated into continuum mechanics.

Recent studies using this new methodology provided exciting results in asymptotic analysis of crack-tip fields. Not only we can find all existing results in this topic, but also new solutions that may be difficult for classical methods. In order to present clearly the application of the new method, let us consider the most classical problem in this topic, a semi-infinite crack in an elastic plane stress body as studied by Williams (1957).

2: HAMILTONIAN SYSTEM WITH RADIAL COORDINATE TREATED AS "TIME"

Consider the domain $\{-\pi \le \theta \le \pi,\ 0 \le r \le R\}$ in the polar coordinate system. The strain energy in this domain can be represented as function of displacements $\{u_r \quad u_\theta\}$:

$$U = \frac{E}{2(1-\nu^2)}\int_{-\pi}^{\pi}\int_{0}^{R}\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + 2\nu\left(\frac{\partial u_r}{\partial r}\right)\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)^2 + \frac{1-\nu}{4}\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)^2\right] r dr d\theta \qquad (4)$$

where E is the Young modular and ν is the Poisson ratio. For more convenience, we perform the following variable changes:

$$r = e^{\xi} \quad \xi = \ln r \qquad (5)$$

Since the radial coordinate ξ is treated as "time", we note $\dfrac{\partial}{\partial \xi} = (\bullet)$, then (4) becomes

$$U=\frac{E}{2(1-\nu^2)}\int_{-\pi}^{\pi}\int_{-\infty}^{R}\left[(\dot{u}_r)^2+2\nu(\dot{u}_r)\left(u_r+\frac{\partial u_\theta}{\partial\theta}\right)+\left(u_r+\frac{\partial u_\theta}{\partial\theta}\right)^2+\frac{1-\nu}{4}\left(\dot{u}_\theta-u_\theta+\frac{\partial u_r}{\partial\theta}\right)^2\right]d\xi d\theta \tag{6}$$

Without taking the external forces into account, the potential energy of the structure Π=U. We first take the displacement components as the generalized displacement vector $\boldsymbol{q}$.

$$\boldsymbol{q}=\{u_r \quad u_\theta\}^T \tag{7}$$

Then $U=U(\dot{\boldsymbol{q}},\boldsymbol{q})$, the principle of minimum potential energy can be written as:

$$\frac{d}{d\xi}\frac{\partial U}{\partial\dot{\boldsymbol{q}}}-\frac{\partial U}{\partial\boldsymbol{q}}=0 \tag{8}$$

Introducing the generalized momentum vector $\boldsymbol{p}=\dfrac{\partial U}{\partial\dot{\boldsymbol{q}}}$, one obtains:

$$\boldsymbol{p}=\begin{Bmatrix}\dfrac{E}{1-\nu^2}\left[\dfrac{\partial u_r}{\partial\xi}+\nu\left(u_r+\dfrac{\partial u_\theta}{\partial\theta}\right)\right]\\ \dfrac{E}{2(1+\nu)}\left[\dfrac{\partial u_\theta}{\partial\xi}-u_\theta+\dfrac{\partial u_r}{\partial\theta}\right]\end{Bmatrix}=r\begin{Bmatrix}\sigma_{rr}\\ \sigma_{r\theta}\end{Bmatrix} \tag{9}$$

By adopting the notation:

$$S_{rr}=r\sigma_{rr}\quad S_{\theta\theta}=r\sigma_{\theta\theta}\quad S_{r\theta}=r\sigma_{r\theta} \tag{10}$$

the generalized momentum vector becomes:

$$\boldsymbol{p}=\{S_{rr}\quad S_{r\theta}\}^T \tag{11}$$

Equation (9) can be rewritten as:

$$\dot{\boldsymbol{q}}=\boldsymbol{A}\boldsymbol{q}+\boldsymbol{D}\boldsymbol{p} \tag{12}$$

with

$$\boldsymbol{A}=\begin{bmatrix}-\nu & -\nu\dfrac{\partial}{\partial\theta}\\ -\dfrac{\partial}{\partial\theta} & 1\end{bmatrix}\quad \boldsymbol{D}=\begin{bmatrix}\dfrac{1-\nu^2}{E} & 0\\ 0 & \dfrac{2(1-\nu)}{E}\end{bmatrix} \tag{13}$$

Now by substituting $\boldsymbol{p}=\dfrac{\partial U}{\partial\dot{\boldsymbol{q}}}$ into (8) and by using the part integration, we have

$$\dot{\boldsymbol{p}}=\boldsymbol{B}\boldsymbol{q}+\boldsymbol{C}\boldsymbol{p} \tag{14}$$

with

$$\boldsymbol{B}=\begin{bmatrix}E & E\dfrac{\partial}{\partial\theta}\\ -\dfrac{\partial}{\partial\theta}(E\,\cdot) & -\dfrac{\partial}{\partial\theta}\left(E\dfrac{\partial}{\partial\theta}\right)\end{bmatrix}\quad \boldsymbol{C}=\begin{bmatrix}\nu & -\dfrac{\partial}{\partial\theta}\\ -\dfrac{\partial}{\partial\theta}(\nu\,\cdot) & -1\end{bmatrix} \tag{15}$$

Equations (12) and (15) form the dual differential equations governing the posed problem. Moreover, it is more convenient to combine the dual variable $\boldsymbol{q}$ and $\boldsymbol{p}$ into a single vector

$$\boldsymbol{v}=\{\boldsymbol{q}^T\quad \boldsymbol{p}^T\}^T \tag{16}$$

Therefore, the dual equations (12) and (15) are written into a single equation:

$$\dot{\boldsymbol{v}}=\boldsymbol{H}\boldsymbol{v} \tag{17}$$

with

$$\boldsymbol{H}=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{D}\\ \boldsymbol{B} & \boldsymbol{C}\end{bmatrix} \tag{18}$$

Equation (17) is the governing equation obtained in the frame of the Hamiltonian system. The boundary conditions at crack lips are:

$$\begin{aligned}&S_{r\theta}(\theta=\pm\pi)=0\\ &\left(u_r+\frac{\partial u_\theta}{\partial\theta}+\frac{\nu S_r}{E}\right)(\theta=\pm\pi)=0\end{aligned} \tag{19}$$

3: HAMILTONIAN SYSTEM WITH ANGULAR COORDINATE TRATED AS "TIME"

Now we take the angular coordinate θ as "time". Then we note $\frac{\partial}{\partial\theta}=\left(\bullet\right)$. By using the variable changes (5), the strain energy (4) becomes

$$U=\frac{1}{2(1-\nu^2)}\int_{-\pi}^{\pi}\int_{0}^{R}\left[\left(\frac{\partial u_r}{\partial\xi}\right)^2+2\nu\left(\frac{\partial u_r}{\partial\xi}\right)\left(u_r+\dot{u}_\theta\right)+\left(u_r+\dot{u}_\theta\right)^2+\frac{1-\nu}{4}\left(\frac{\partial u_\theta}{\partial\xi}-u_\theta+\dot{u}_r\right)^2\right]d\xi d\theta \quad (20)$$

By following the same procedure as described in the precedent paragraph, we obtain the following system of first-order differential equations:

$$\begin{Bmatrix}\dot{u}_r\\ \dot{u}_\theta\\ \dot{S}_{r\theta}\\ \dot{S}_\theta\end{Bmatrix}=\begin{bmatrix}0 & 1-\partial/\partial\xi & 2(1+\nu)/E & 0\\ -1-\nu\partial/\partial\xi & 0 & 0 & (1-\nu^2)/E\\ -E\partial^2/\partial\xi^2 & 0 & 0 & 1-\nu\partial/\partial\xi\\ 0 & 0 & -(1+\partial/\partial\xi) & 0\end{bmatrix}\begin{Bmatrix}u_r\\ u_\theta\\ S_{r\theta}\\ S_\theta\end{Bmatrix} \quad (21)$$

We write (21) under concise form:

$$\dot{\boldsymbol{v}}=\boldsymbol{H}\boldsymbol{v} \quad (22)$$

where

$$\boldsymbol{v}=\left\{\boldsymbol{q}^T \quad \boldsymbol{p}^T\right\}$$
$$\boldsymbol{q}=\left\{u_{rr} \quad u_{\theta\theta}\right\}^T \quad \boldsymbol{p}=\left\{S_{r\theta} \quad S_{\theta\theta}\right\}^T \quad (23)$$
$$\boldsymbol{H}=\begin{bmatrix}0 & 1-\partial/\partial\xi & 2(1+\nu)/E & 0\\ -1-\nu\partial/\partial\xi & 0 & 0 & (1-\nu^2)/E\\ -E\partial^2/\partial\xi^2 & 0 & 0 & 1-\nu\partial/\partial\xi\\ 0 & 0 & -(1+\partial/\partial\xi) & 0\end{bmatrix}$$

Here we have found the governing equations under the conventional form in the Hamiltonian system. The boundary conditions can be written in a simple form. For the present problem, we have:

$$\boldsymbol{p}(\theta=\pi)=\boldsymbol{p}(\theta=-\pi)=\boldsymbol{0} \quad (24)$$

The governing equation (22) and the boundary conditions (24) allows the solution of the posed problem.

4: THEOREM OF EXPANSION

One can use the method of the separation of variables to solve the governing equations (17) and (22) established in the frame of the Hamiltonian system. We assume that the variable vector $\boldsymbol{v}$ can be written under separable form:

$$\boldsymbol{v}=\exp(\mu\xi)\boldsymbol{\psi}(\theta) \quad (25)$$

where μ is an undetermined eigenvalue, $\boldsymbol{\psi}(\theta)$ is the corresponding eigenvector depending only on θ. The following theorem of expansion has been demonstrated by Zhong (1997):

$\mu=0$ is a multiple eigenvalue of the problem. If μ is an eigenvalue, $-\mu$ is also one. Let $\boldsymbol{\psi}^0_k$, $\boldsymbol{\psi}_i$, and $\boldsymbol{\psi}_{-i}$ be the corresponding eigenvectors respectively. Then any vector $\boldsymbol{v}$ in the solution space can be represented as linear combination of the eigenvectors:

$$\boldsymbol{v}=\sum_{i=1}^{\infty}\left(a_i\exp(\mu_i)\boldsymbol{\psi}_i+b_i\exp(-\mu_i)\boldsymbol{\psi}_{-i}\right)+\sum_{k=1}^{n}a_k^0\boldsymbol{\psi}_k^0 \quad (26)$$

where a_i, b_i and a^0_k are the undermined constants.

$$a_i = -\frac{\exp(-\mu_i \xi)\int_{-\pi}^{\pi} \boldsymbol{\psi}_{-i}^T \boldsymbol{J}\boldsymbol{v} d\theta}{\int_{-\pi}^{\pi} \boldsymbol{\psi}_{-i}^T \boldsymbol{J}\boldsymbol{\psi}_i d\theta} \quad b_i = \frac{\exp(\mu_i \xi)\int_{-\pi}^{\pi} \boldsymbol{\psi}_i^T \boldsymbol{J}\boldsymbol{v} d\theta}{\int_{-\pi}^{\pi} \boldsymbol{\psi}_i^T \boldsymbol{J}\boldsymbol{\psi}_{-i} d\theta} \qquad (27)$$

$$with \quad \boldsymbol{J} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

5: SOLUTION METHODS

We believe there exist many methods, analytical or numerical, to solve the governing equations (17) and (22). We only present here a very explicit method to solve the equation (22). By introducing (25) into (22), we find:

$$\dot{\boldsymbol{\psi}} = \boldsymbol{H}(\mu)\boldsymbol{\psi} \qquad (28)$$

with

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1-\mu & 2(1+\nu)/E & 0 \\ -1-\nu\mu & 0 & 0 & (1-\nu^2)/E \\ -E\mu^2 & 0 & 0 & 1-\nu\mu \\ 0 & 0 & -(1+\mu) & 0 \end{bmatrix} \qquad (29)$$

It is interesting to remark that H is a constant matrix with a parameter μ. For more complicated cases H may be function of θ. In fact, the solution of (28) can be written under following analytical form:

$$\boldsymbol{\psi}(\theta) = \exp\left(\int_{\theta_0}^{\theta} \boldsymbol{H}(\mu) d\theta\right)\boldsymbol{\psi}(\theta_0) \qquad (30)$$

For the present problem, H is a constant matrix, $\theta_0=-\pi$, (30) becomes:

$$\boldsymbol{\psi}(\theta) = \exp\left(\boldsymbol{H}(\mu)(\theta+\pi)\right)\boldsymbol{\psi}(-\pi) \qquad (31)$$

The eigenvalue μ can be determined by the boundary conditions (24). For $\theta=\pi$, we can write:

$$\boldsymbol{\psi}(\pi) = \exp\left(\boldsymbol{H}(\mu)2\pi\right)\boldsymbol{\psi}(-\pi) \qquad (32)$$

Let $\boldsymbol{G} = \exp\left(\boldsymbol{H}(\mu)2\pi\right)$, equation (32) can be rewritten as follows:

$$\begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix}(\theta=\pi) = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix}(\theta=-\pi) \qquad (33)$$

According to the boundary conditions (24), we have immediately:

$$\det\left|\boldsymbol{G}_{21}\right| = 0 \qquad (34)$$

This is the equation allowing determination of the eigenvalue μ. Iterative procedures can be used in solving (34). Once μ obtained, the vector $\boldsymbol{q}(\theta=\pi)$ can be calculated from the second equation of (33). Then the variable vector $\boldsymbol{\psi}(\theta)$ can straightforwardly be computed from equation (31).

6: MULTI-MATERIAL PROBLEMS

One of the advantages of the approach taking the angular coordinate as "time" is its high capacity to deal with the multi-material problems. Consider a crack formed from several homogeneous elastic materials. The material 1 occupies the sectorial domain $[\theta_0,\theta_1]$, named zone 1; the material 2 occupies the zone 2, bounded by $[\theta_1,\theta_2]$, and so on. This is a general configuration of this class of problems. The cases like a crack lying at an interface between two dissimilar materials or a crack meeting an interface with an arbitrary angle can be regarded as its special cases. We adopt the superscript $^{(i)}$ to indicate the quantities in the zone i , for example, $v^{(i)}$, $H^{(i)}$, etc.. For this problem, one can establish the governing equation (22) for each zone:

$$\dot{\boldsymbol{v}}^{(i)} = \boldsymbol{H}^{(i)}\boldsymbol{v}^{(i)} \qquad (34)$$

The boundary conditions at the two free surfaces of the crack are:

$$\begin{aligned} \boldsymbol{p}^{(1)}(\theta=-\pi) &= \boldsymbol{0} \\ \boldsymbol{p}^{(n)}(\theta=\pi) &= \boldsymbol{0} \end{aligned} \qquad (35)$$

The continuity conditions across the interfaces are:

$$\begin{aligned} &\boldsymbol{v}^{(1)}(\theta=\theta_1)=\boldsymbol{v}^{(2)}(\theta=\theta_1)\\ &\cdots\cdots\\ &\boldsymbol{v}^{(n-1)}(\theta=\theta_{n-1})=\boldsymbol{v}^{(n)}(\theta=\theta_{n-1}) \end{aligned} \tag{36}$$

These relations show the advantage of the choice of the angle coordinate treated as "time": the multi-material problem can be dealt with as a single material problem since the variable vector *v* is continuous across all the interfaces. This makes the resolution of the governing equation (34) much easier.

According to the basic solution (30), we can write the relationship between $\boldsymbol{\psi}^{(i)}(\theta_i)$ and $\boldsymbol{\psi}^{(i)}(\theta_{i-1})$ for each zone:

$$\boldsymbol{\psi}^{(i)}(\theta_i)=\exp\left(\boldsymbol{H}^{(i)}(\mu)(\theta_i-\theta_{i-1})\right)\boldsymbol{\psi}^{(i)}(\theta_{i-1}) \tag{37}$$

Let $\boldsymbol{G}^{(i)}=\exp\left(\boldsymbol{H}^{(i)}(\mu)(\theta_i-\theta_{i-1})\right)$. According to the continuity conditions across the interfaces, we write immediately the relationship between $\boldsymbol{\psi}^{(n)}_{\cdot}(\pi)$ and $\boldsymbol{\psi}^{(1)}_{\cdot}(-\pi)$, namely:

$$\begin{aligned} &\boldsymbol{\Psi}^{(n)}(\pi_i)=\boldsymbol{G}\boldsymbol{\Psi}^{(1)}(-\pi)\\ &\boldsymbol{G}=\prod_{i=n}^{1}\boldsymbol{G}^{(i)} \end{aligned} \tag{38}$$

The eigen-equation is identical as equation (34).

7: CONCLUSIONS

In this paper, we have made a rough draft on the introduction of the Hamiltonian system into the asymptotic analysis in fracture mechanics. We insist on the fact that this approach is not only a new formalism other the traditional methods, but also a real powerful tool to study problems in this topic. By using this new approach, we have successfully investigated some problems that may be difficult for traditional Airy function method (for example, asymptotic analysis for a plane crack formed by several anisotropic materials; asymptotic analysis for cracks in a bending plate formed by several anisotropic materials; residual stress analysis of a crack in elastic-plastic materials etc.). We believe that a large domain can be found in applying this new approach into fracture mechanics.

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