# BEAM THEORY AND WEIGHT FUNCTION METHODS FOR MODE I DELAMINATION WITH LARGE SCALE BRIDGING

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## ABSTRACT

A nonlinear fracture mechanics model is formulated for analysis of mode I delamination of orthotropic double cantilever beam specimens in the presence of large scale bridging conditions. The model accounts for the presence of regions of contact along the wake of the crack, which may form due to the action of the bridging mechanisms. The problem is solved using a nonlinear integral equation approach in terms of stress intensity factors at the crack tip. An approximate weight function is proposed and validated numerically for a pair of concentrated forces acting on the surfaces of the delamination. The model is applied to investigate the influence of the orthotropy of the material on the fracture behavior and the validity of approximated solutions based on beam theory.

**KEYWORDS:** Nonlinear fracture mechanics, weight functions, delamination, strengthening mechanisms, anisotropic material, large scale bridging.

## INTRODUCTION

Mode I and mixed mode delamination in large scale bridging conditions, such as those created by through thickness reinforcement in composite laminates, shows unusual phenomena of crack face closure, crack arrest and crack propagation with crack face contact, which have no precedent in the delamination of conventional tape laminates [1,2]. In [3] the authors considered a typical mixed mode geometry, the Mixed Mode Bending specimen proposed by Crews and Reeder, and explained these phenomena by means of a simple analytical model based on Timoshenko beam theory. The model treats the delaminated arms of the specimen as beams on an elastic, generally nonlinear, foundation of Winkler type with the constitutive laws of the springs given by the bridging law, which characterizes the bridging mechanism. The crack closure phenomenon is a manifestation of the oscillations of the function representing the deflection of the beams in the wake of the crack. The wavelength of the function, $\lambda$, sets the characteristic length scale of the problem, which in the case of linear bridging mechanisms is given approximately by $\lambda/4 = \pi/2 \sqrt[4]{4k_d / \beta_3}$ , with $\beta_3$ the modulus of the foundation and $k_d$ the flexural stiffness of the beam cross section. Once the limit configuration for crack tip closure is approached, the fracture response of the specimen will depend on the geometry and the loading conditions. In the case of a specimen symmetric about its midplane and in the absence of mode II loading, the crack will stop and the specimen will break by mechanisms other then delamination. In the presence of mode II loading or in asymmetric specimens, the crack will continue to propagate and the propagation will be opposed not only by the bridging mechanism but also by friction acting in the regions of contact.

The model proposed in [3] explains qualitatively all the problems associated with mixed mode large scale bridging delamination. However, the model makes strong assumptions which could affect the solutions quantitatively, namely it schematizes the specimen as a one-dimensional structure, it neglects the influence of the elastic material in front of the crack (built-in ends assumption) and it deals only approximately with regions of contact between the delaminated faces and the effect these regions may have on crack propagation driven by mode II loading.

In this paper a nonlinear fracture mechanics model is formulated for analysis of delamination crack growth which removes the above mentioned assumptions, assumes a two-dimensional deformation field and accounts for the orthotropy of the material. The problem is solved through an integral equation approach in terms of stress intensity factors at the crack tip. Since the crack closure phenomenon is controlled by the mode I response of the laminate, focus in this initial work is restricted to this problem.

## FRACTURE PARAMETERS IN ORTHOTROPIC DOUBLE CANTILEVER BEAMS

### Stress intensity factors

An exact solution for the stress intensity factor $K_{IP}$ due to a pair of concentrated forces $P$ applied per unit width onto the crack faces of a double cantilever beam at a distance $d$ from the crack tip has been obtained by Foote and Buchwald [4]. They solved the problem by applying the Wiener-Hopf technique to an isotropic, arbitrarily loaded infinite strip and representing the concentrated loads in terms of the Dirac delta function. A simple formula approximating the exact solution, which has an accuracy of 1.1% and can be applied to double cantilever beams with an uncracked ligament $c > 2h$, is also given in [4]:

$$\frac{K_{IP}h^{0.5}}{P} = \sqrt{12}\left(\frac{d}{h}+0.673\right)+\sqrt{\frac{2h}{\pi d}}-\left[0.815\left(\frac{d}{h}\right)^{0.619}+0.429\right]^{-1} \quad (1)$$

where $h$ is the half thickness of the specimen. For $d/h \geq 0.3$ the exact $K_{IP}$ is well represented (error always lower than 4%) by Gross and Srawley's boundary collocation solution, approximately given by the first bracket term on the right hand side. The same limit solution is given by the modified beam theory of Kanninen, which removes the assumption of built-in ends to account for the elasticity of the uncracked ligament. For very large $d/h$, Eq. (1) approaches the elementary beam theory solution of a double cantilever beam with built-in ends, $K_I h^{0.5}/P = \sqrt{12}\, d/h$ [3]. For very small $d/h$ the dimensionless $K_{IP}$ of Eq. (1) approaches Irwin's solution for a semi-infinite crack in an infinite sheet, $K_I h^{0.5}/P = (2/\pi\, h/d)^{0.5}$. A lower limit for the normalized crack length $a/h$ of the double cantilever beam specimen must be set for Eq. (1) to be valid for all $d/h$, $0 < d/h \leq a/h$. Irwin's solution for very small $d/h$ is correct only if $a >> d$, and should be replaced by Tada's solution [5] for a finite crack of length $a$ in a semi-infinite sheet when $a/h$ also becomes very small. A conservative lower limit for $a/h$ can be set as $a/h \geq 0.3$, so that when $d/h = a/h = 0.3$, the beam theory solution is already approached.

The stress intensity factor due to a pair of concentrated forces $P_i$ applied on the crack faces of an orthotropic double cantilever beam at the coordinate $x_1 = x_{1i}$ (Fig. 1.a) can be deduced from the expression of the strain energy release rate $\mathcal{G}_I$ obtained by Suo et al. [6] making use of the orthotropic relationship $K_{Ii} = \sqrt{\mathcal{G}_I E_1^{'}}$. Plane stress conditions are assumed along with a principally orthotropic material with $E_1^{'} = (\sqrt{2E_1E_3}\lambda^{1/4})/\sqrt{1+\rho}$, $\lambda = E_3/E_1$ and $\rho = \sqrt{E_1E_3}/2G_{13} - \sqrt{\nu_{13}\nu_{31}}$ the orthotropic ratios [6], $E_1$ and $E_3$ the Young's moduli in the $x_1$ and $x_3$ directions, $G_{13}$ the shear modulus and $\nu_{13}$ and $\nu_{31}$ Poisson's ratios. The dimensionless stress intensity factor is then given by:

$$\frac{K_{Ii}h^{0.5}}{P_i} = \frac{\lambda^{3/8}}{\sqrt{n}}\sqrt{12}\left(\frac{a-x_{1i}}{h}+Y_I(\rho)\lambda^{-1/4}\right) \quad (2)$$

where:

$$Y_I(\rho) = 0.677+0.146(\rho-1)-0.0178(\rho-1)^2+0.00242(\rho-1)^3 \quad (3)$$

and $n = \sqrt{(1+\rho)/2}$. For an isotropic material ($\lambda = \rho = 1$), $Y_I(\rho) = 0.677$ and Eq. (2) coincides with Gross and Srawley's solution. The last term on the right hand side of eq. (2) describes the influence of the elasticity of the uncracked ligament ahead of the crack tip and it vanishes for large $(a-x_{1i})/h$, when the solution for an orthortropic beam with built-in ends is recovered. Equation (2), has 0.5% accuracy for all $(a-x_{1i})/h \geq 2\,\lambda^{-1/4}$ and $0 \leq \rho \leq 4$.

An exact solution for the stress intensity factor $K_{Ii}$ when $(a-x_{1i})/h < 2\lambda^{-1/4}$ is not available in the literature. However, the method of orthotropy rescaling, the examination of Eqs. (1) and (2) and the observation that Irwin's solution for very small $d/h = (a- x_{1i})/h$ maintains its validity also in orthotropic sheets [7], suggest the following formula [10]:

$$\frac{K_{Ii}h^{0.5}}{P_i} = \frac{\lambda^{3/8}}{\sqrt{n}}\sqrt{12}\frac{(a-x_{1i})}{h}\left(1+Y_I(\rho)\lambda^{-1/4}\frac{h}{(a-x_{1i})}\right)+\sqrt{\frac{2h}{\pi(a-x_{1i})}} - \left[0.815\left(\frac{0.677}{Y_I(\rho)\sqrt{n}}\lambda^{1/4}\frac{a-x_{1i}}{h}\right)^{0.619}\lambda^{-1/8}+\frac{\sqrt{n}}{\lambda^{1/8}Y_I(\rho)\sqrt{12}}\right]^{-1} \quad (4)$$

which has the right asymptotic behaviors for large and small $(a-x_{1i})/h$. The validity of Eq. (4) for intermediate values of $(a-x_{1i})/h$ has been checked through finite element calculations for a range of $\lambda$ and $\rho$ typical of composite laminates, $0.025 \leq \lambda \leq 1$ and $0 \leq \rho \leq 5$, and the relative error has been found to be always lower than 2%. The lower limit for the normalized crack length for which Eq. (4) is valid can be defined by referring to the limit for an isotropic material and exploiting orthotropic rescaling of lengths [6], which yields $a/h \geq 0.3\,\lambda^{-1/4}$ with a 4% error in the case $\rho = 1$. When a reason exists for studying very small cracks as well as non-small cracks in numerical work, Eq. (4) can be combined very easily with Tada's result for the appropriate domains of $a/h$.

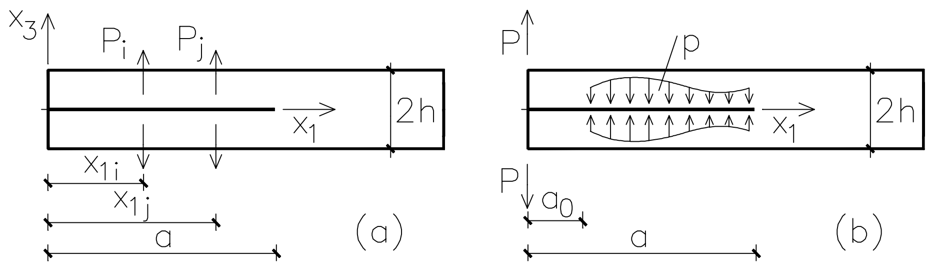


Figure 1: Schematic of the DCB specimen under different loading conditions.

**Crack opening displacement**

The crack opening displacement $u_3$ at the coordinate $x_{1i}$ due to a pair of opening forces $P_j$ acting at $x_{1j}$, Fig. 1, is obtained from the localized compliance $\lambda_{ij} = u_3(x_{1i}) / P_j$ which can be defined through an energy balance or Castigliano's theorem as shown in [8] for an isotropic body. The localized compliance is given by:

$$\lambda_{ij} = \frac{u_3(x_{1i})}{P_j} = \frac{2}{E_1'}\int_0^a \frac{K_{Ii}(a,x_{1i})K_{Ij}(a,x_{1j})}{P_iP_j}\,da \quad (5)$$

where $E_1'$ is the orthotropic constant defined above, $P_i$ is a pair of fictitious forces acting at $x_{1i}$, and $K_{Ii}$ and $K_{Ij}$ are the stress intensity factors at the crack tip due to $P_i$ and $P_j$, respectively, given by Eq. (4), [10].

**ORTHOTROPIC DOUBLE CANTILEVER BEAM WITH LARGE SCALE BRIDGING**

The stress intensity factor at the crack tip of a double cantilever beam with tractions $p$ acting along the bridged portion of the crack as shown in Fig. 1.b is given by:

$$K_{\rm I} = K_{\rm IP} + K_{\rm Ip} = K_{\rm IP} - \int_{a_0}^{a} \frac{K_{\rm Ii}(a, x_{1\rm i})}{P_{\rm i}} p[u_3(x_{1\rm i})] {\rm d}x_{\rm i} \tag{6}$$

where $a_0$ is the unbridged length of the crack, $K_{IP}$ is the stress intensity factor due to the external loads $P_i$ acting at $x_{1i} = 0$, and $K_{Ip}$ is the stress intensity factor due to opening tractions $p$; $K_{Ii}/P_i$ represents the Green's function of the problem and obtained from Eq. (4). The tractions $p$ depend on the crack displacement and are a priori unknown in Eq. (6). If $u_3(x_{1i}) > 0$, then $p[u_3(x_{1i})] = p_3[u_3(x_{1i})]$ is the closing traction developed by the bridging mechanisms. The value of $p_3$ as a function of $u_3$ is defined through the bridging traction law, $p_3(u_3)$, which is one of the data of the model. If $u_3(x_{1i}) = 0$, then $p[u_3(x_{1i})] = - p_c[(x_{1i})]$ is the opening traction depicting the effect of the contact pressure. The contact pressure and the size of the regions of contact are unknown a priori and can be determined through a compatibility condition for the crack opening displacement.

The crack opening displacement $u_3(x_{1i})$ is obtained by applying the superposition principle and Eq. (6):

$$u_3(x_{1\rm i}) = u_3(x_{1\rm i})_{\rm P} + u_3(x_{1\rm i})_{\rm p} = \lambda_{\rm iP} P - \int_{a_0}^{a} \lambda_{\rm ij} p[u_3(x_{1\rm j})]\, {\rm d}x_j \tag{7}$$

which yields:

$$\begin{aligned} \frac{u_3(x_{1\rm i})}{h} = & \frac{2P}{E_1^{'}h} \int_0^{a/h} \frac{K_{\rm IP}(a/h)h^{0.5}}{P} \frac{K_{\rm Ii}(a/h, x_{1\rm i}/h)h^{0.5}}{P_{\rm i}} {\rm d}(\frac{a}{h}) \\ & - \frac{2}{E_1^{'}} \int_{a_0/h}^{a/h} \int_{\max[x_{1\rm i}/h, x_{1\rm j}/h]}^{a/h} \frac{K_{\rm Ii}(a/h, x_{1\rm i}/h)h^{0.5}}{P_{\rm i}} \frac{K_{\rm Ij}(a/h, x_{1\rm j}/h)h^{0.5}}{P_{\rm j}} {\rm d}(\frac{a}{h})\, p[u_3(x_{1\rm j}/h)]\, {\rm d}(\frac{x_{1\rm j}}{h}) \end{aligned} \tag{8}$$

Note that the dimensionless $K_I$'s appearing in Eq. (8) and in the equations that follow depend also on the orthotropic ratios, $\lambda$ and $\rho$, as shown in Eq. (4).

**Crack propagation in large scale bridging**

At the onset of crack propagation the crack tip stress intensity factor of Eq. (6) is equal to the intrinsic fracture toughness, $K_I = K_{Ic}$, and the dimensionless critical load for crack propagation takes the form:

$$\frac{P_{\rm cr}}{K_{\rm Ic}h^{0.5}} = \frac{1}{\frac{K_{\rm IP}(a/h)h^{0.5}}{P}} \left\{ 1 + \frac{p_{30}h^{0.5}}{K_{\rm Ic}} \int_{a_0/h}^{a/h} \left[ \frac{K_{\rm Ii}(a/h, x_{1\rm i}/h)h^{0.5}}{P_{\rm i}} \right] \frac{p[u(x_{1\rm i}/h)]}{p_{30}} {\rm d}(\frac{x_{1\rm i}}{h}) \right\} \tag{9}$$

where $p_{30}$ is a normalizing value of the crack face tractions, $p_3$, given for instance by their maximum value. The dimensionless number on the right hand side of Eq. (9), $p_{30}h^{0.5}/K_{\rm Ic}$, is a measure of the brittleness of the structure. Recalling the expression for $E_1^{'}$ and that $K_{Ic} = \sqrt{\mathcal{G}_{\rm Ic}E_1^{'}}$, Eq. (9) can be modified to allow direct comparison between isotropic and othotropic cases:

$$\frac{P_{\rm cr}}{\sqrt{\mathcal{G}_{\rm Ic}E_1 h}} = \frac{\lambda^{3/8}}{\sqrt{n}} \frac{1}{\frac{K_{\rm IP}(a/h)h^{0.5}}{P}} \left\{ 1 + \frac{p_{30}h^{0.5}}{\sqrt{\mathcal{G}_{\rm Ic}E_1}} \int_{a_0/h}^{a/h} \left[ \frac{K_{\rm Ii}(a/h, x_{1\rm i}/h)h^{0.5}}{P_{\rm i}} \right] \frac{p[u(x_{1\rm i}/h)]}{p_{30}} {\rm d}(\frac{x_{1\rm i}}{h}) \right\} \tag{10}$$

The normalized crack opening displacement at the generic coordinate $x_{1i}$ is obtained substituting $P = P_{cr}$ into Eq. (8):

$$\frac{u_3(x_{1\mathrm{i}})E_1'}{K_{\mathrm{Ic}}h^{0.5}} = 2\frac{P_{\mathrm{cr}}}{K_{\mathrm{Ic}}h^{0.5}} \int_0^{a/h} \frac{K_{\mathrm{IP}}(a/h)h^{0.5}}{P} \frac{K_{\mathrm{Ii}}(a/h, x_{1\mathrm{i}}/h)h^{0.5}}{P_{\mathrm{i}}} \mathrm{d}(\frac{a}{h})$$

$$- 2\frac{p_{30}h^{0.5}}{K_{\mathrm{Ic}}} \int_{a_0/h}^{a/h} \int_{\max[x_{1\mathrm{i}}/h, x_{1\mathrm{j}}/h]}^{a/h} \frac{K_{\mathrm{Ii}}(a/h, x_{1\mathrm{i}}/h)h^{0.5}}{P_{\mathrm{i}}} \frac{K_{\mathrm{Ij}}(a/h, x_{1\mathrm{j}}/h)h^{0.5}}{P_{\mathrm{j}}} \mathrm{d}(\frac{a}{h}) \frac{p[u(x_{1\mathrm{j}}/h)]}{p_{30}} \mathrm{d}(\frac{x_{1\mathrm{j}}}{h}) \quad (11)$$

The statically indeterminate problem defined by the nonlinear integral equations (9) and (11) is solved for general bridging laws, $p_3(u_3)$, through a discretization. A self-consistent solution for the crack profile is obtained iteratively through a numerical procedure following the approach of [8,9].

**Dugdale type bridging law, $p_3 = p_{30}$**

In the special case of bridging mechanisms described by a Dugdale type bridging law, $p_3 = p_{30}$, Eqs. (9) and (11) simplify and Eq. (9) alone gives the dimensionless critical load for crack propagation. Beam theory predicts the absence of regions of contact for this case and this qualitative characteristic is confirmed by the more accurate calculations of the integral equation approach. Setting aside for this paper the interesting question of the nature of contact regions when they do occur (e.g., for linear bridging laws), a detailed assessment is made here of the limitations of elementary beam theory for predicting crack propagation in the presence of large scale bridging.

Figures 2.a, 2.b and 2.c show dimensionless diagrams of the critical load for crack propagation as a function of the normalized crack length in a double cantilever beam specimen with $a_0 = 0$. Three different values of $p_3$ are considered, as marked. The curves named (*a*), (*b*) and (*c*) in each diagram describe the response of an isotropic material, an orthotropic material with $\lambda = 0.1$ and $\rho = 3$ (e.g. a graphite epoxy laminate) and an orthotropic material with $\lambda = 0.05$ and $\rho = 5$ (e.g. a boron-epoxy laminate), respectively. The dashed curves depict the elementary beam theory solution (built-in ends, negligible shear deformations). The dotted curve in Fig. 2b, obtained using Timoshenko beam theory for an isotropic material and marked as $\gamma \neq 0$, highlights the influence of the shear deformations.

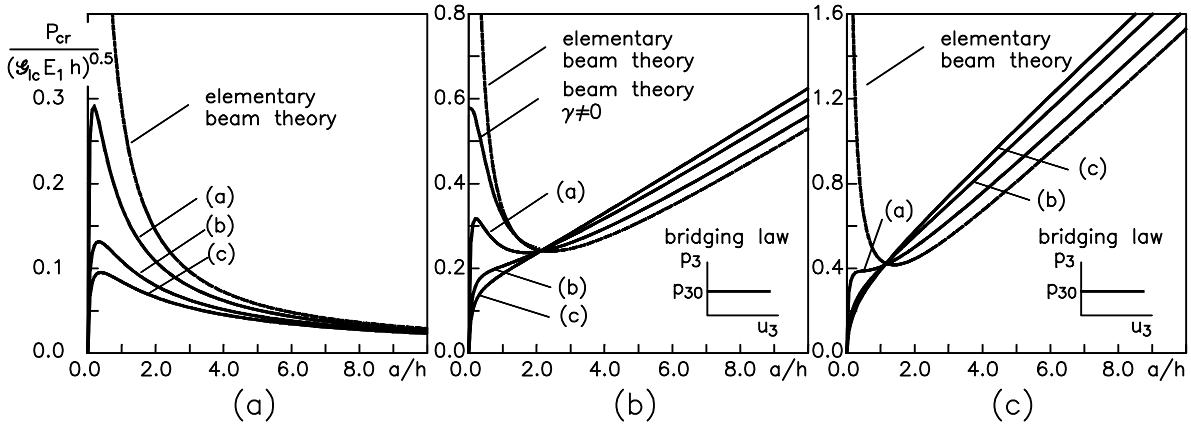


Figure 2: Dimensionless critical load versus normalized crack length in orthotropic DCB specimens. (a) No bridging. (b) Bridging tractions $p_3 = 0.1\sqrt{\mathcal{G}_{\mathrm{Ic}}E_1/h}$ . (c) Bridging tractions $p_3 = 0.3\sqrt{\mathcal{G}_{\mathrm{Ic}}E_1/h}$ .

Figure 2.a confirms the validity of elementary beam theory in unreinforced specimens when *a/h* is sufficiently high. The anisotropy of the material affects the response only for relatively small values of *a/h*. The influence of the anisotropy of the material on the structural response apparently seems to be more marked in members reinforced through the thickness (Figs. 2.b and 2.c). In this case, two different regimes of behavior are delineated by a transition value of $a/h = 1/(1.73p_{30}\sqrt{\mathcal{G}_{\mathrm{Ic}}E_1/h}\,)^{0.5}$, corresponding to the point where all curves cross each other. If *a/h* is smaller than the transition value, the anisotropy of the material strongly affects the response and the elementary beam theory solution does not describe the actual behavior even qualitatively. For *a/h* larger than the transition value, all curves tend to become parallel with a common slope given by $1/2\ p_{30}\sqrt{\mathcal{G}_{\mathrm{Ic}}E_1/h}$ and the deviation between the correct solution and the elementary beam

theory solution becomes independent of the crack length and given by $1/2Y_I(\rho)\lambda^{-1/4}\, p_{30}\sqrt{\mathcal{G}_{Ic}E_1/h}$ . However, the fractional error is $Y_I(\rho)\lambda^{-1/4}/(a/h)$, which is independent of the intrinsic fracture toughness of the laminate, $\mathcal{G}_{Ic}$, and the magnitude of the bridging tractions and coincides with the analogous fractional error of the case with no bridging. It depends only the crack length and the degree of anisotropy. This error is due to the assumption of neglecting the influence of the elastic material ahead of the crack tip and could be removed by using a modified beam theory (Kanninen's, Williams's).

## CONCLUSIONS

An approximate weight function has been proposed and validated numerically for a pair of point forces acting on the surfaces of a delamination crack in a possibly thin orthotropic body. The weight function allows mode I large scale bridging problems in beams and plates to be formulated as integral equations without the limitations imposed on accuracy by beam theory approximations. In particular, the crack tip singularity will be properly represented. The integral equations can be solved using well-known, computationally efficient and accurate methods. The weight function strongly depends on the anisotropy ratio. This is a feature of the plate or beam geometry.

The first application of the new weight function to the problem of a large zone of uniform bridging tractions (the Dugdale bridging model) shows the ranges of crack lengths over which beam theories of different order succeed and fail. The presence of large scale bridging is found not to significantly increase the sensitivity of the solutions to the degree of anisotropy with respect to the case with no bridging.

While elementary beam theory will always be correct for sufficiently large crack lengths, $a/h$, there is a regime of small crack lengths, $a/h \lesssim 2$, where rigorous solutions are required, e.g. based on integral equation methods. Moreover, there is a regime of practical interest for laboratory specimens, $2 \lesssim a/h \lesssim 10$, where elementary beam theory yields only qualitatively correct trends and solutions based on integral equation methods or on modified beam theory are required for quantitative accuracy.

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