ORAL REFERENCE: ICF10-083OR

AN ANALYTICAL TECHNIQUE FOR STUDYING INTERACTING BRANCHED CRACKS IN A PLATE

Stephanie C. TerMaath[1] and S. Leigh Phoenix[2]

[1] Department of Civil Engineering, Cornell University,
Ithaca, NY 14853 USA
[2] Theoretical and Applied Mechanics, Cornell University,
Ithaca, NY 14853 USA

ABSTRACT

An analytical method for studying interacting branched cracks in an infinite plate is developed. Based on superposition and dislocation theory, this method can be used to determine the full stress and displacement fields in a cracked material. In addition, stress singularities at both crack tips and wedges (created by crack branching) are calculated so that crack growth and initiation can be analyzed at all locations of possible crack propagation. A key concept of the method is the development of dislocation distributions that represent the opening displacements and capture the physical behavior of the cracks. Each distribution is a shaping series representing characteristic crack behaviors; therefore, development of effective distributions is a crucial aspect of this work. Branched cracks of complex shapes under general loading conditions can be evaluated with this method. Results show rapid convergence for few degrees of freedom (as measured by the number of dislocation distribution terms included in a particular analysis).

KEYWORDS

Dislocation distribution, superposition, stress intensity factor, branched crack

INTRODUCTION

Material imperfections, corrosion, and fatigue loading can create conditions that cause cracks to branch or grow in such a way that they have multiple crack tips. Damage zones containing cracks of such complex shapes, specifically many interacting branched cracks, pose a challenging problem when attempting to evaluate these areas for potential crack propagation and possible failure. To address this type of fracture, a two-dimensional analytical technique has been developed to study interacting branched cracks in an infinite plate. Based on superposition and dislocation theory, this method can be used to determine full stress and displacement fields in addition to stress intensity factors at crack tips and branch locations for a cracked plate. Previous researchers have used similar approaches to study these types of cracks [1-6], and an extensive review of this area of research has been performed [6].

OVERVIEW OF THE ANALYTICAL TECHNIQUE

To calculate the stress and displacement fields in an infinite plate containing an array of cracks of complex shape, each crack's opening displacement profile must be determined such that all crack faces are traction-free under the given loading conditions. (An opening displacement profile is the shape of a deformed crack.) Once the opening displacement profiles are known, this solution can also be used to determine the stress intensity factors at the crack tips and branch locations in order to study crack propagation. Superposition is applied at the global and local levels, and a dislocation distribution approach is utilized, to solve for the opening shapes of an existing crack array. Several excellent texts on this subject are available in the literature [7-9].

Superposition

To solve this boundary value problem, superposition is first applied at the global level by modeling the cracked plate as two separate problems (the trivial problem and the auxiliary problem) where the sum of their solutions equals the solution to the original problem. The trivial problem consists of the given plate under the specified far field loading but without the cracks. Meanwhile, the auxiliary problem is the given cracked plate, but without the far field loading. The loading conditions for the auxiliary problem are instead prescribed tractions applied to the crack faces that are calculated to be equal and opposite to the stresses induced in the uncracked material at the location of the crack faces. This loading insures that the crack faces are traction-free in the original problem when the stress field solutions to the trivial and auxiliary problems are summed. Obtaining a solution to the auxiliary problem, which constitutes the bulk of the analytical and computational effort, requires the development and superposition of certain solutions on the local level reflecting detailed crack geometric features.

To solve for the opening displacement profiles of the auxiliary problem, the first step is to subdivide cracks into a series of straight crack segments spanning from branch point to tip. For example, the branched crack of Figure 1 is divided into three crack segments, each with its own local coordinate system. Once the opening displacement profile for a single segment is determined, its effects on the full stress field can be evaluated separately from the other crack segments. Superposition of the local solutions for all of the respective crack segments yields the full solution to the auxiliary problem.

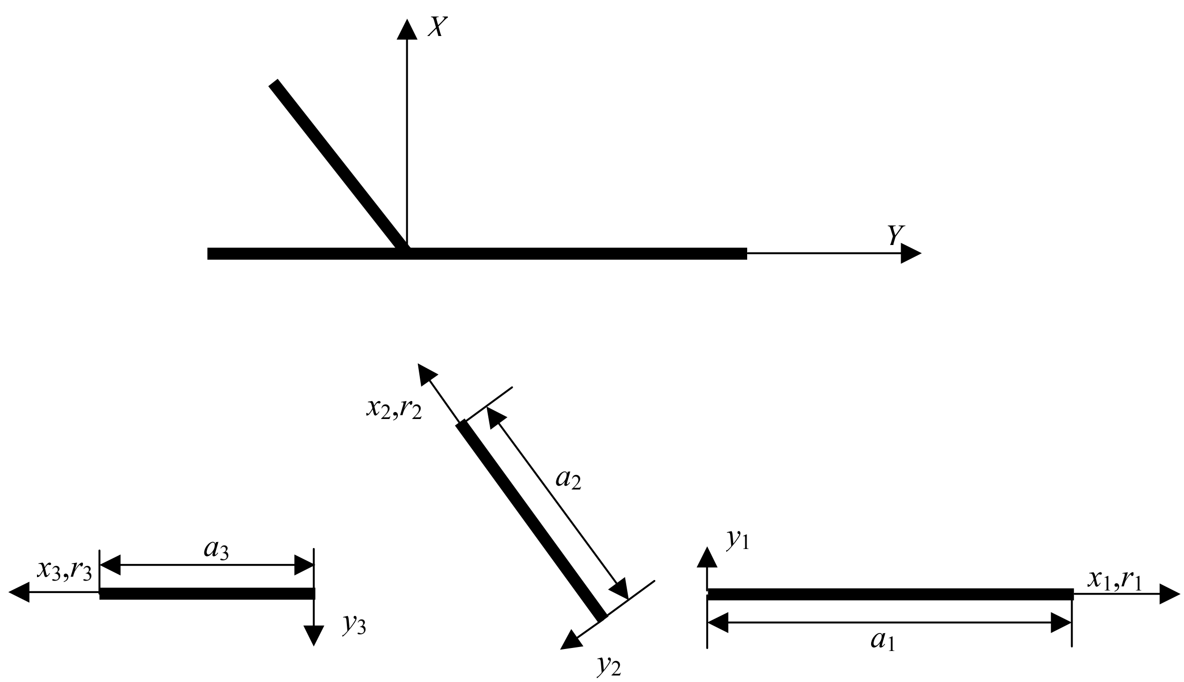

Figure 1: Global and local coordinate systems for a branched crack

Dislocation Distributions

Dislocation distributions are the means of describing the opening displacement profile of a crack segment and inducing the prescribed crack face tractions of the auxiliary problem. A dislocation distribution, $\mu_\eta(r)$, is defined as the derivative of a crack segment's opening displacement profile, where r is an axis coincident with the crack segment. To determine the stresses induced at a point (x,y) in the material caused by all of the crack segments, the individual effects of each must first be determined.

Consider crack segment i acting alone (as though all other crack segments are closed) in an infinite, linearly elastic, isotropic plate with local coordinate system (x_i, y_i) such that the x_i-axis lies along the crack segment which has length a_i. The distance along the x_i-axis is r_i and is measured from the origin. The dislocation distributions for a single crack segment are symbolized as $\mu_{1i}(r_i)$ and $\mu_{2i}(r_i)$. The subscripts 1 and 2 represent the tangential and normal directions respectively. The stress components caused by this individual crack segment at point (x,y) are written in terms of a complex variable formulation as

$$
\begin{aligned}
s_{xy}^{(i)} &= -\frac{2G}{\pi(1+\kappa)}\left\{ y\,\mathrm{Re}\left(Z_2^2\right) + \mathrm{Re}\left(Z_1^1\right) + y\,\mathrm{Im}\left(Z_2^1\right) \right\} \\
s_{yy}^{(i)} &= -\frac{2G}{\pi(1+\kappa)}\left\{ \mathrm{Re}\left(Z_1^2\right) - y\,\mathrm{Im}\left(Z_2^2\right) + y\,\mathrm{Re}\left(Z_2^1\right) \right\} \\
s_{xx}^{(i)} &= -\frac{2G}{\pi(1+\kappa)}\left\{ \mathrm{Re}\left(Z_1^2\right) + y\,\mathrm{Im}\left(Z_2^2\right) + 2\,\mathrm{Im}\left(Z_1^1\right) - y\,\mathrm{Re}\left(Z_2^1\right) \right\}
\end{aligned} \tag{1}
$$

where G is the shear modulus of the material, ν is Poisson's ratio, and κ is Kosolov's constant (3-4ν, for plane strain and (3-ν)/(1+ν), for plane stress). These stresses are symbolized by s to indicate that they are created by a single crack segment and are oriented in its local coordinate system. Note that point (x,y) must also be converted to the local coordinate system for use in these equations. The full stress field due to all crack segments will be denoted by σ and is determined by summing the contributions from all individual crack segments after they are converted to the global coordinate system. The Z are Cauchy singular integrals to be evaluated in closed form in terms of the dislocation distributions and are given as

$$
\begin{aligned}
Z_1^\eta &= \int_0^{a_i} \frac{\mu_{\eta i}(r_i)\mathrm{d}r_i}{z - r_i} \\
Z_2^\eta &= \int_0^{a_i} \frac{\mu_{\eta i}(r_i)\mathrm{d}r_i}{\left(z - r_i\right)^2} = -\frac{\mathrm{d}}{\mathrm{d}z} Z_1^\eta
\end{aligned} \tag{2}
$$

where $z = x + \mathrm{i}y$ and η = 1 or 2 referring to the tangential and normal directions respectively. For the cases where the point (x,y) falls along the crack segment, these integrals are evaluated as Cauchy Principal Value Integrals. Solutions to these integrals for given dislocation distributions can be found in [6].

The stress equations are functions of unknown dislocation distributions for the various crack segments. These dislocation distributions are approximated by summing together different types of series that each captures a fundamental crack or wedge behavior (such as singularities at branch locations and tips). The Cauchy singular integrals are evaluated analytically for each term of these series. The results from each particular term are then multiplied by an unknown weighting coefficient (or degree of freedom). Therefore, the stress equations for each crack segment are now captured through simple algebraic equations of unknown weighting coefficients.

Satisfying The Traction-Free Condition

Physical conditions dictate that the crack faces are traction-free in the full problem. To ensure this condition, the opening displacement profiles for each crack segment in the auxiliary problem must be exactly those caused by the prescribed tractions. Therefore, a series of equations to enforce traction-free crack faces in the tangential and normal directions is applied simultaneously at a given set of points along each crack segment. These equations take the form

$$\begin{aligned} \sigma_{xy}^{\infty} n_y + \sigma_{xx}^{\infty} n_x &= -n_y \sum_{i=1}^{N} s_{xy}^{(i)} - n_x \sum_{i=1}^{N} s_{xx}^{(i)} \\ \sigma_{yy}^{\infty} n_y + \sigma_{xy}^{\infty} n_x &= -n_y \sum_{i=1}^{N} s_{yy}^{(i)} - n_x \sum_{i=1}^{N} s_{xy}^{(i)} \end{aligned} \tag{3}$$

where N is the total number of crack segments. The left hand side of these equations represents the tractions induced at the crack faces by the loading conditions, while the right hand side represents the tractions caused by the opening displacements (dislocation distributions) of the crack segments. Also, n_x and n_y are the X and Y components, respectively, of the normal to the bottom (-) crack faces. The σ^{∞} are the far field stresses applied to the plate in the directions denoted by their subscripts.

Solving for the Unknown Coefficients

Satisfying the traction-free condition along the crack faces (Eqn. 3) at a suitably chosen set of points results in a system of equations. These equations are linear functions of the unknown weights of each term from each series. To calculate the weights a large matrix must be inverted; therefore, the use of efficient and physically realistic series is imperative to reduce the number of degrees of freedom to the smallest number possible. Solving this set of simultaneous equations requires the inversion of a large matrix. Selection of points and number of terms produces an over-determined matrix that is solved by a least squares fit. Once the weighting coefficients have been calculated, stress and displacement fields and stress intensity factors can be readily determined [6].

OPENING DISPLACEMENT SERIES

Different types of series (wedge, tip, and polynomial) are used to build the opening displacement profiles of the cracks. Emphasis was placed on creating efficient series to capture all necessary types of physical behavior while minimizing the number of degrees of freedom in an analysis. Wedge series based on singular eigenvalues [10-12] calculated at material wedges greater than 180° induced by crack branching will not be presented, since the example provided does not include a wedge of this type. It should also be noted that constraint equations are enforced at branch points to eliminate mathematical, but non-physical singularities, created by adjoining crack segments [6]. Each term of a series is multiplied by an unknown weighting coefficient, c, and each series is used independently in both the tangential and normal modes. Furthermore, each type of series must be applied to every crack segment.

Polynonial Series

Polynomial series, $P(r)$, provide flexibility in manipulating the overall opening displacement shape in addition to allowing for translation and rotation at branch locations. This series is formulated to constrain non-physical jump opening displacements and slopes at the tip end of a crack segment and takes the form

$$P(r) = \sum_{j=0}^{n-2} c_{jp} \left(\left(\frac{r}{a}\right)^{j} - (n-j)\left(\frac{r}{a}\right)^{(n-1)} + (n-j-1)\left(\frac{r}{a}\right)^{n} \right) \tag{4}$$

Tip Series

Tip series, $T(r)$, incorporate the ½ singularity and higher order behavior at crack tips. This series is developed to avoid non-physical jump opening and slope behavior at branch locations and is written as

$$T(r) = \sum_{j=0}^{n-2} c_{jt} \left(\left(\frac{a-r}{a} \right)^{\frac{2j+1}{2}} - (n-j) \left(\frac{a-r}{a} \right)^{\frac{2(n-1)+1}{2}} + (n-j-1) \left(\frac{a-r}{a} \right)^{\frac{2n+1}{2}} \right) \qquad (5)$$

BRANCHED CRACK EXAMPLES

Rigorous testing for accuracy was performed using results of other researchers [13-16], and agreement was achieved in all cases studied [6]. Results available in the literature provided only stress intensity factors at crack tips, so this parameter formed the basis of the comparisons. However, overall results with this method demonstrated rapid convergence in terms of weighting coefficients, stress intensity factors, and tractions along crack faces as induced by the computed opening displacement profiles.

This method can be applied to branched cracks of any configuration and crack segment lengths. Cracks need not be symmetric nor limited to a certain number of crack tips or growth directions. Furthermore, loading is not restricted by type and can be any combination of shear and normal loading modes. As an example, results from one parameter study of two interacting branched cracks are provided.

For this particular case, two symmetric interacting branched cracks in an infinite plate under unit biaxial loading were evaluated. Branch segments were of unit length while the main crack segment had a length of 2. The branch angle, β, and the separation distance, d, were varied. Calculated values for K_I at the inner crack tips are shown in graphical form in Figure 3 as a representative sample of the results. Note that as the distance, d, was increased, the K_I values converged to those of a single, isolated, branched crack.

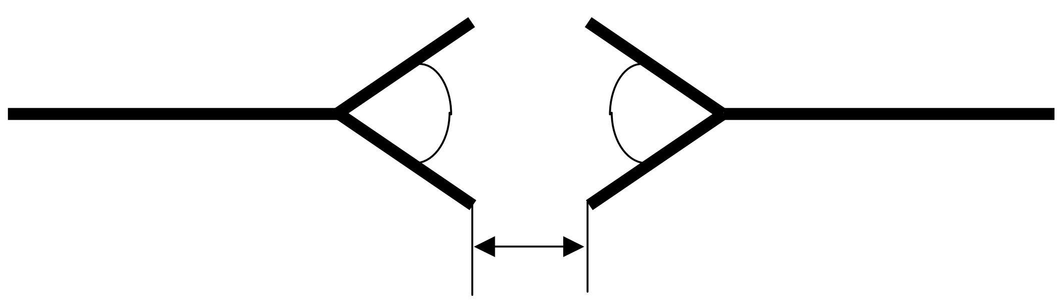

Figure 2: Two interacting branched cracks under unit biaxial loading

ACKNOWLEDGEMENTS

This work was performed under a fellowship from the American Association of University Women.

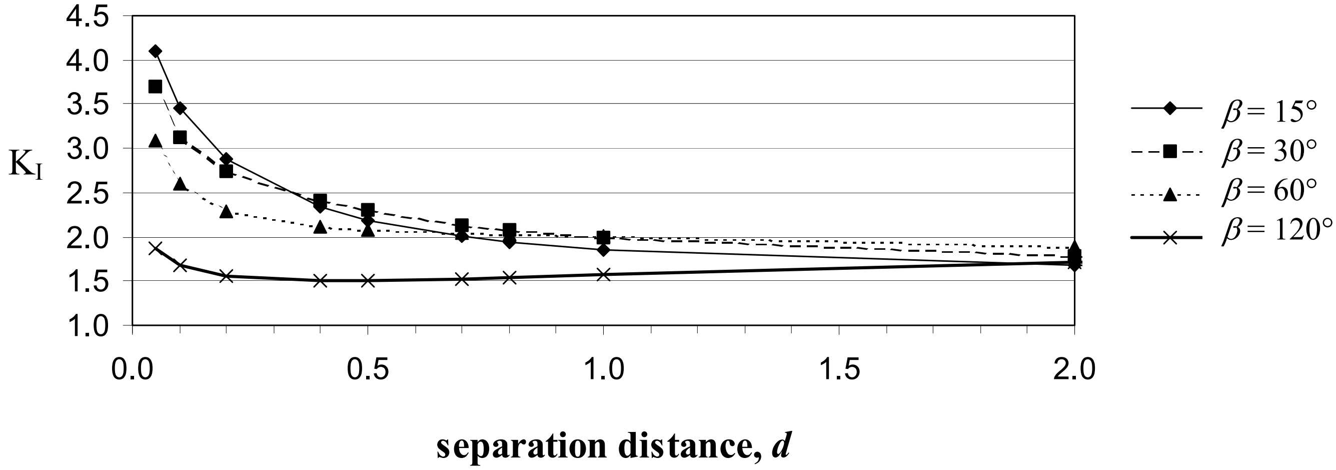

Figure 3: Mode I stress intensity factors for varying separation distances and branch configurations

REFERENCES

1. Vitek, V. (1977) *Int. J. Fract.* 13, 481.
2. Lo, K.K. (1978) *J. Appl. Mech.* 45, 797.
3. Niu, J. and Wu, M.S. (1997) *Eng. Fract. Mech.* 57, 665.
4. Burton, Jr., J.K. and Phoenix, S.L. (2000) *Int. J. Fract.* 102, 99.
5. TerMaath, S.C. and Phoenix, S.L. (2000) In: *Fatigue and Fracture Mechanics: 31st Volume, ASTM STP 1389*, 331, G.R. Halford and J.P. Gallagher (Eds). American Society for Testing and Materials, West Conshohocken.
6. TerMaath, S.C. (2000) Dissertation, Cornell University, USA.
7. Lardner, R. (1974) *Mathematical theory of dislocations and fracture*. University of Toronto Press, Great Britain.
8. Hirth, J. and Lothe, J. (1982) *Theory of dislocations*. John Wiley & Sons, New York.
9. Hills, D.A., Kelly, P.A., Dai, D.N., and Korsunsky, A.M. (1996) *Solution of crack problems: the distributed dislocation technique*. Kluwer Academic Publishers, Dordrecht.
10. Timoshenko, S.P. and Goodier, J.N. (1970) *Theory of Elasticity*, Third Edition, McGraw-Hill, New York.
11. Barber, J.R. (1992) *Elasticity*. Kluwer Academic Publishers, Boston.
12. Williams, M.L. (1952) *J. Appl. Mech.* 19, 526.
13. Theocaris, P.S. (1972) *J. Mech. Phys. Solids* 20, 265.
14. Kitagawa, H. and Yuuki, R. (1975) *Trans. Japan Soc. Mech. Engrs.* 41-346, 1641.
15. Isida, M. and Noguchi, H. (1983) *Trans. Japan Soc. Mech. Engrs.* 49-440, 469.
16. Chen, Y.Z. and Hasebe, N. (1995) *Eng. Fract. Mech.* 52, 791.