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A LIMIT EQUILIBRIUM OF QUASIBRITTLE MATERIALS WITH THIN INCLUSIONS

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ABSTRACT

A new effective approach to estimate a tensile strength of materials with inhomogeneities was proposed. As the example, the calculation of ultimate strength for cast irons with graphite inclusions or inclusions of phosphide eutectic was carried out. The comparison with the known experimental data was done and good coincidence was shown.

KEYWORDS

Inclusion, stressed state, plane strain, displacements, stress intensity factors, ultimate strength.

INTRODUCTION

The most of constructional materials are heterogeneous. They comprise cracks, cavities or impurities as the stress concentrators. Therefore, theoretical ultimate strength determination of materials in view of the presence of structural defects is an important scientific and technical problem.

PROBLEM DEFINITION

The elastic isotropic quasibrittle material with small volume content of structural defects is considered. We model such material by the infinite elastic body with the isolated cylindrical inclusion. It is assumed that G is the shear modulus, μ - Poisson coefficient for basic material (named as matrix), G_1 is the shear modulus for inclusion and μ_1 - its Poisson coefficient (the case of plane strain is supposed). Let's choose the system of rectangular Cartesian coordinates so that axis Oy coincides with a cylindrical axis of inclusion, and coordinate axes compound the right ternary. The inclusion cross section is described by the equation $z = \pm h(x)$, where $|x| \le a$, $|z| \le c$, $\lambda = a/c >> 1$, a and c are the semiaxes of cylinder. It is assumed, that during deformation the inclusion is rigidly linked to a base material. At infinity, the body is loaded by uniformly tensile forces p along z-axis. The problem is to determine the value $p = p_*$, for which the local fracture of a matrix or inclusion or separation process is begun.

MATHEMATICAL MODEL OF ELASTIC INCLUSION DEFORMATION

Using the model relationships [1], we obtain the following correspondences between stresses and displacements on the surfaces of the thin elastic inclusion under given loading:

$$\frac{2}{1-\mu_1}\frac{\partial}{\partial x}\left(u_x^1\right)_* + \frac{\mu_1}{G_1(1-\mu_1)}\left(\sigma_{zz}^1\right)_* + \frac{1}{G_1 h(x)}\int_{-a}^{x}\left[\sigma_{xz}^1\right]_* dt = 0\,,$$

$$\frac{\left(\sigma_{zz}^1\right)_*}{G_1} - \frac{2\mu_1}{1-2\mu_1}\frac{\partial}{\partial x}(u_x^1)_* - \frac{2(1-\mu_1)}{1-2\mu_1}\cdot\frac{[u_z^1]_*}{h(x)} = 0\,, \tag{1}$$

where the symbols $[\ ]_*$ and $(\ )_*$ describe the jumps and sums of a function on passing through the surfaces of inclusion, i.e. $[A]_* = A^+ - A^-$, $(A)_* = A^+ + A^-$, $A^{\pm} = A|_{z=\pm h(x)}$, u_x^1, u_z^1 are the components of a displacements vector in inclusion; $\sigma_{xz}^1, \sigma_{zz}^1$ - are the constituents of a vector of stresses inside a defect. Eqn. 1 can describe all kinds of elastic inclusion deformations. If $G_1 = 0$ or $G_1 \to \infty$ the dependences for a cavity or an absolutely rigid inclusion respectively can be obtained from Eqn. 1. The system of dependences forms the mathematical model of elastic inclusion deformation.

STRESSED STATE DETERMINATION IN A BODY WITH INCLUSION

At first we present [2] the elastic problem for a body with thin inclusion as a composition of two problems: the problem a) for the homogeneous body under the given applied external loading inducing the vector of stresses $\vec{\sigma}_z^o\left(\sigma_{xz}^o,\ \sigma_{zz}^o\right)$, and the problem b) for the body with the cavity $\{z = \pm h(x), -\infty < y < \infty\}$ subjected to unknown stresses $\tilde{\vec{\sigma}}_z = -\vec{\sigma}_z^o + \vec{\sigma}_z^1$ on inclusion surfaces. We represent the displacement vector as the sum $\vec{u}\,(u_x, u_z) = \tilde{\vec{u}}\,(\tilde{u}_x, \tilde{u}_z) + \vec{u}^{\,o}\,(u_x^o, u_z^o)$. Using the supposition about thinness of inclusion we can replace [1] a task b) by the singular problem c) for the body with the mathematical cut $\{|x| \le a,\ -\infty < y < \infty\}$ with the stresses $\tilde{\vec{\sigma}}_z\,(\tilde{\sigma}_{xz},\ \tilde{\sigma}_{zz})$ applied to it surfaces.

The solution of problem a) is known:

$$[u_z^o]_* = p(1-\mu)h(x)/G; \qquad (\sigma_{zz}^o)_* = 2p\,. \tag{2}$$

Using Fourier integral transformation the solution of the problem c) can be obtained [3] in kind of such dependences concerning the stress and displacement jumps on the inclusions surfaces.

$$(\tilde{\sigma}_{zz})_* = \frac{G}{\pi(1-\mu)}\int_{-a}^{a}\frac{[\tilde{u}_z]_*'}{t-x}dt + \frac{1-2\mu}{2\pi(1-\mu)}\int_{-a}^{a}\frac{[\tilde{\sigma}_{xz}]_*}{t-x}dt;$$

$$\left(\tilde{u}_x\right)'_* = \frac{1-2\mu}{2\pi(1-\mu)}\int_{-a}^{a}\frac{[\tilde{u}_z]_*'}{t-x}dt - \frac{3-4\mu}{4\pi G(1-\mu)}\int_{-a}^{a}\frac{[\tilde{\sigma}_{xz}]_*}{t-x}dt\,. \tag{3}$$

Here and further symbol prime means the derivate on *x*.
Substituting Eqn. 3 to Eqn. 1 we obtain the system of singular integro-differential equations for unknown vectors $[\tilde{\vec{\sigma}}_z]_*$ and $[\tilde{\vec{u}}]_*$ in such form

$$B_{11}\int_{-a}^{a}\frac{[\tilde{u}_z]_*'}{t-x}dt + B_{12}\int_{-a}^{a}\frac{[\tilde{\sigma}_{xz}]_*}{t-x}dt + B_{13}\int_{-a}^{a}[\tilde{\sigma}_{xz}]_* dt = -M_1;$$

$$B_{21}\int_{-a}^{a}\frac{[\tilde{u}_z]_*'}{t-x}dt + B_{22}\int_{-a}^{a}\frac{[\tilde{\sigma}_{xz}]_*}{t-x}dt - B_{23}\frac{[\tilde{u}_z]_*}{h(x)} = -M_2\,, \quad z = 0,\ \ |x| \le a, \tag{4}$$

where $B_{11}, B_{12}, ..., B_{23}$ - are known [4] coefficients depended by the elastic modules of inclusion; $M_1,\ M_2$ - are known functions defined by the solution of task a). When we'll solve Eqn. 4 we calculate the stress intensity factor K_I for the problem c) using the expression [1]

$$K_I = -\lim_{x \to -a} \sqrt{2\pi(a-x)} \left\{ \frac{G}{2(1-\mu)} [\tilde{u}_z]'_* + \frac{1-2\mu}{4(1-\mu)} [\tilde{\sigma}_{xz}]_* \right\}. \tag{5}$$

For a finding the stress distribution in basic material near the inclusion we use the formula [5]

$$\sigma_{zz} = \frac{2(\rho + \tilde{x})K_I}{\sqrt{\pi(\rho + 2\tilde{x})^3}} + \frac{\rho\sqrt{\rho}}{\sqrt{(\rho + 2\tilde{x})^3}} \tilde{\sigma}_{xx} + \sigma_{zz}^o ; \qquad \tilde{x} = x - a . \tag{6}$$

Here ρ is the radius of curvature in top of defect; $\tilde{\sigma}_{xx}$ are the end stresses inside inclusion defined [5] by the stresses jump $[\tilde{\sigma}_{xz}]_*$

$$\tilde{\sigma}_{xx}(x) = -\frac{1}{2h(x)} \int_{-a}^{x} [\tilde{\sigma}_{xz}]_* \, dt . \tag{7}$$

The normal stresses inside inclusion we obtain by means of Eqns. 2 and 3.

A LOCAL FRACTURE CRITERION

Using the first theory of strength we receive that in body a limit equilibrium state will be occur if even one of values the stress in a body near inclusion or inside defect else the stress on a intermediate contact surface attain their ultimate strength, i.e.

$$max \left\{ \frac{\sigma(p)}{\sigma_B^{(m)}};\ \frac{\sigma^1(p)}{\sigma_B^1};\ \frac{\sigma^{(c)}(p)}{\sigma_B^{(c)}} \right\} \Bigg|_{p=p_*} = 1 , \tag{8}$$

where $\sigma(p),\ \sigma^1(p),\ \sigma^{(c)}(p)$ are maxima of stresses in a matrix, inside inclusion and on contact, respectively; $\sigma_B^{(m)}, \sigma_B^1, \sigma_B^{(c)}$ are values of corresponding ultimate strength.

SOLUTIONS OF APPLIED PROBLEMS

Using the results of pre-previous paragraph we'll obtain such formulae for stress concentration and stresses inside defect in the case of isolated elliptical $\left(h(x) = \sqrt{a^2 - x^2} \big/ \lambda \right)$ tunnel inclusion in infinite body.

Analysis of Eqn. 6 shows that in the case $G_1 < G$ maximum of stresses σ_{zz} is attained at points $x = \pm a$. Then

$$max\, \sigma_{zz} = \sigma_{zz}\big|_{x=\pm a} = 2K_I \big/ \sqrt{\pi\rho} + \tilde{\sigma}_{xx}(a) + \sigma_{zz}^o ,$$

or for elliptical inclusion ($\rho = a/\lambda^2$) we find

$$\sigma_{zz}\big|_{x=\pm a} = p\,(1 + \Theta) , \tag{9}$$

$$\sigma^1_{zz} = (1-\mu)\varepsilon p \frac{\varepsilon(\lambda(3-2\mu)+2(1-\mu))+\lambda(1-2\mu_1+2\lambda(1-\mu_1))}{\omega}, \qquad \varepsilon = \frac{G_1}{G}, \tag{10}$$

where designations were accepted as

$$\omega = 2\varepsilon(1-\mu)(1-\mu_1)(1+\lambda^2)+\lambda[\varepsilon^2(3-4\mu)-2\varepsilon\mu_1(1-2\mu)+1-2\mu_1)], \tag{11}$$

$$\Theta = \lambda\,\omega^{-1}\{2\lambda[\varepsilon\mu(\mu_1-\varepsilon\mu)+1-2\mu_1-\varepsilon(1-\mu-\mu_1)]+\varepsilon(1-\mu)[3-2\mu_1-\varepsilon(3-2\mu)]\}. \tag{12}$$

In the case $G_1 \geq G$ maximum of stresses σ_{zz} is attained at point $\tilde{x}_1 = -[3\rho\sqrt{\pi\rho}\,\tilde{\sigma}_{xx}(a)+4\rho K_I]/(2K_I)$ (see Eqn. 6) and it is equal to

$$max\,\sigma_{zz} = \sigma_{zz}\big|_{\tilde{x}=\tilde{x}_1} = \frac{2}{3\sqrt{3\pi\rho}}\sqrt{-\frac{K_I^3}{\sqrt{\pi\rho}\,\tilde{\sigma}_{xx}+K_I}}+\sigma^o_{zz}. \tag{13}$$

By means of Eqns. 8, 9, 10 and 13 the value of p_* can be calculated for given materials.

In the case of quasibrittle material with a great volume content of inclusions we shall use the model of infinite body with double periodical system of elliptic cylindrical inclusions, see Figure 1. Thus, at first we solve a problem for a periodic system of coplanar inclusions in a body. In this case the kern of Eqn. 4 $L(t,x)=1/(t-x)$ was replaced by the kern $L_1(t,x,d_1)=\frac{\pi}{d_1}ctg\frac{\pi(t-x)}{d_1}$ and the problem solution was obtained by a little parameter decomposition method.

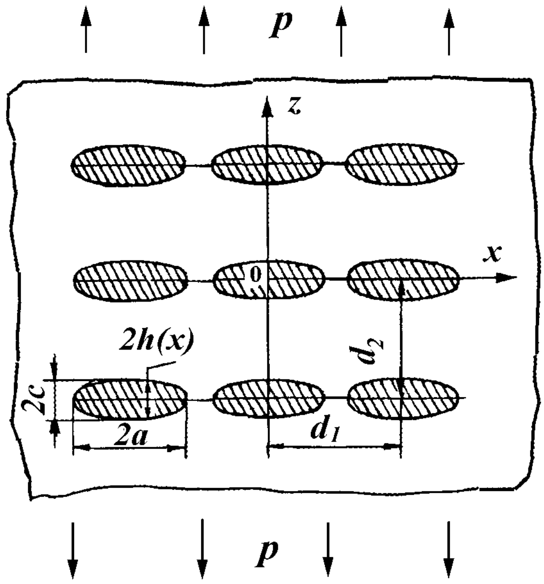

Figure 1: The schema of material with the great volume content of inclusions

Then we solve an elastic problem for a body with periodical system of parallel elliptical cylindrical inclusions. In this case the kern $L(t,x)=1/(t-x)$ of Eqn. 4 will be replaced by the kern $L_2(t,x,d_2)=\frac{\pi}{d_2}\left[2cth\frac{\pi(t-x)}{d_2}-\frac{\pi(t-x)}{d_2}cosech^2\frac{\pi(t-x)}{d_2}\right]$. The solution of problem was obtained by the similar way. Thus using the method of boundary interpolation [6] we find the solution of the problem for a body with double periodical system of associated inclusions. If we shall assume that a quasibrittle matrix damages first of all (as experiments show) we get the values of the tensile strength for such material with the associated inclusions by means of Eqn. 8:

$$\sigma_B = \sigma_B^{(m)} \big/ \Phi\,(\varepsilon, \lambda, \alpha_1, \alpha_2), \tag{14}$$

where

$$\Phi\,(\varepsilon,\lambda,\alpha_1,\alpha_2) = 1 + \Theta\,[1 + F\,(\beta,\alpha_1,\alpha_2)]\,, \qquad \alpha_1 = 2a/d_1\,; \qquad \alpha_2 = 2a/d_2\,;$$

$$F(\beta,\alpha_1,\alpha_2) = \frac{\alpha_1^2 - 3\alpha_2^2}{1+\beta}\cdot\frac{\pi^2}{3\cdot 2^3} - \frac{[(15\beta^2 + 52\beta + 57)\alpha_1^4 + 15(5\beta^2 + 20\beta + 27)\alpha_2^4]\cdot\pi^4}{(1+\beta)^2(3+\beta)\cdot 45\cdot 2^7} +$$

$$+\frac{[(210\beta^4 + 1877\beta^3 + 5597\beta^2 + 7495\beta + 4125)\cdot\alpha_1^6 - 35(42\beta^4 + 409\beta^3 + 1433\beta^2 + 2149\beta + 1785)\cdot\alpha_2^6]}{(1+\beta)^3(3+\beta)(5+\beta)}\cdot$$

$$\cdot\frac{\pi^6}{945\cdot 2^{10}} + o(\alpha_1^6, \alpha_2^6)\,; \qquad \beta = \frac{2\,\varepsilon\,\lambda\,(1-\mu)(1-\mu_1)}{1 - 2\mu_1 - \varepsilon\mu_1(1-2\mu)}, \quad \varepsilon < 1\,. \tag{15}$$

The comparison the theoretical values of tensile strength obtained by Eqn. 14 with famous [7] experimental data for cast-irons with graphite inclusions was carried out. Experimental results were obtained for different cast-iron alloys with 4% mass content of carbon and diverse forms of graphite inclusions – from circular to laminar mode (see symbols ∇, □, o at Figure 2). It is easy to convince somebody that there is a close correspondence between calculated and experimental results.

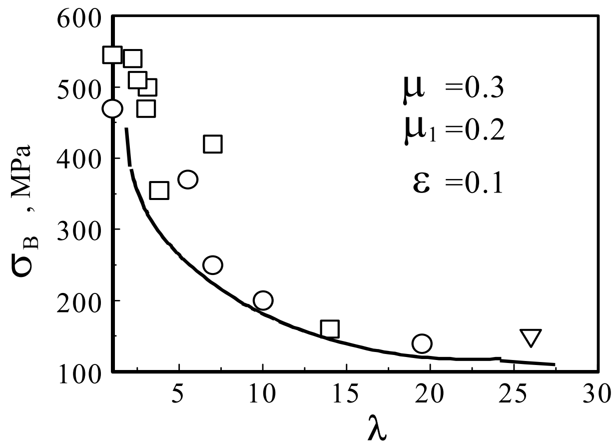

Figure 2: The comparison of theoretical tensile strength determination and experimental data for grey cast-iron with graphite inclusions

We also elaborated the estimation of tensile strength values for cast-irons with various mass content of phosphorus. At increasing a content of phosphorus the phosphide eutectic inclusions are formed. Experiments for grey cast-irons with phosphide eutectic were carried out at Technical University of Zaporizhzhya (Ukraine) under the leadership of prof. Volchok I.P. We made a comparison between the theoretical results and experimental data by means of such values of parameters: $\sigma_B^1 = 470\,MPa,\quad \mu_1 = 0.1,\ G_1 = 165\,GPa,\ \mu = 0.25,\ G = 80\ MPa,\ \lambda = 1 \div 10$. A close correspondence between calculated and experimental results was obtained once again.

CONCLUSIONS

1. A mathematical model of elastic inclusion deformation of arbitrary relative rigidity was proposed.
2. A stressed state determination of a body with thin elastic inclusion was carried out.
3. A fracture criterion for quasibrittle materials with inhomogeneities was stated.

4. Using the methods of boundary interpolation and a little parameter decomposition the stress concentration in a material with double periodical system of associated elliptic inclusions was calculated.
5. The formula for the tensile strength determination of quasibrittle materials with great volume content of inclusions was proposed.
6. The comparison the theoretical values of tensile strength for cast-irons with graphite inclusions or the inclusions of phosphide eutectic with the known experimental data were carried out. A close correspondence between them was attained.

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