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A COMPLEX VARIABLE FORMULATION FOR THE INTERACTIONS BETWEEN RIGID INCLUSIONS AND CRACKS

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ABSTRACT

This paper summarizes the recent formulation by Wang and Chau [1] on a new boundary element method (BEM) in terms of complex variables for plane elastic bodies containing cracks, holes and rigid inclusions subjected to mixed displacement/ traction boundary conditions. A complex boundary function $H(t)$, which is a linear combination of the boundary traction and boundary displacement density, is introduced. The present Boundary Integral Formulation can be related directly to Muskhelishvili's formalism. Singular interpolation functions of order $r^{-1/2}$ (where r is the distance measured from the crack tip) are introduced such that singular integrand involved at the element level can be integrated analytically. The interaction between a rigid circular inclusion and a crack is investigated in details. Our results for the stress intensity factor are comparable with those given by Erdogan and Gupta [2] and Gharpuray et al. [3] for a crack emanating from a stiff inclusion, and with those by Erdogan et al. [4] for a crack in the neighborhood of a stiff inclusion.

KEYWORDS

Crack, Circular inclusions, Boundary element method, Complex variable

INTRODUCTION

In recent years, boundary element method (BEM) has widely been applied in solving linear elastic problems and fracture mechanics problems, and has been developed into a powerful numerical technique. In the traditional approach, boundary integral equations are derived by from the Somigliana's identity (e.g. Rizzo [5]; Cruse [6]; Lachat and Watson [7]; Brebbia [8]). The application of BEM has been focused mainly on traction boundary value problems (BVPs), and there is relatively few BEM studies on solving mixed BVPs, to which displacements and tractions may be prescribed on disjoint portions or on the same segment of boundary but along different directions (e.g. Bonnet [9]; Gaul and Schanz [10]). Note, however, that the so-called "mixed BVPs" are sometimes simply referred as BVPs regardless of whether displacement or traction is prescribed on the boundary. A typical example of the mixed BVPs is the interaction between rigid circular inclusions and cracks in plane elastic bodies. There is no BEM that has been proposed in the literature for such problems. Therefore, Wang and Chau [1] recently proposed a robust BEM to solve this problem. This conference paper will present and summarize the main findings by Wang and Chau [1]. For the case of interactions between non-rigid circular inclusions and cracks, we refer to the works of Wang et al. [11], Erdogan et al. [4], Erdogan and Gupta [3], Isida and Noguchi [12], and Gharpuray et el. [3].

The present formulation closely resembles the Muskhelishvili [13-14] formalism. For mixed BVPs formulation in complex variables, we refer to the works by Sherman [15-18] and Lu [19]. These formulations, however, do not originate from the Somigliana's identity and, thus, their relationship to the classical BIE formulation is unclear. But this missing link between these formulations and the usual BEM was considered by Wang and Chau [1].

The main obecjtive of the present paper is to summarize the main findings by Wang and Chau [1]. The new BIE formulation originates from the Somigliana's identity and involves singular integrals of Cauchy type. The present BIE formulation is of the same mathematical form as that derived by Chau and Wang [20]. Thus, the numerical implementation proposed by Wang and Chau [21] will be adopted here for our BEM formulation. One main advantage of the present "complex" variable formulation over the traditional "real" variable formulations (e.g. Ghosh et al. [22]; Bonnet [23]; Frangi and Novati [24]) is that the kernal functions involved in the boundary integral equations are much simpler and, as shown by Wang and Chau [21], they can be dealt with analytically.

BOUNDARY INTEGRAL FORMULATION IN COMPLEX VARIABLE

By consider a two-dimensional linear isotropic elastic body containing m holes and n cracks of arbitrary shape under plane condition (see Figure 1), Chau and Wang [20] derived the following boundary integral formulation for stresses and displacements in terms of a complex unknown function $H(t)$:

$$\sigma_{11}+\sigma_{22}=2[\Phi(z)+\overline{\Phi(z)}], \qquad (z=x_1+ix_2\in\Omega \ \ and \ \ i=\sqrt{-1}), \tag{1}$$

$$\sigma_{22}-\sigma_{11}+2i\sigma_{12}=2[\bar{z}\Phi'(z)+\Psi(z)], \qquad (\bar{z}=x_1-ix_2), \tag{2}$$

$$2G(u_1+iu_2)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}, \tag{3}$$

where

$$\Phi(z)=\frac{1}{4}(s_1^\infty+s_2^\infty)+iC+\frac{1}{2pi}\int_{S+\Gamma}\frac{H(t)dt}{t-z}, \tag{4}$$

$$\Psi(z)=\frac{1}{2}(s_2^\infty-s_1^\infty+2is_{12}^\infty)-\frac{1}{2pi}\int_{S+\Gamma}\left[\frac{\overline{H(t)}-\overline{q(t)}}{t-z}e^{-2ia(t)}+\frac{\bar{t}H(t)}{(t-z)^2}\right]dt, \tag{5}$$

$$j(z)=\int\Phi(z)dz=\frac{1}{4}(s_1^\infty+s_2^\infty)z+iCz-\frac{1}{2pi}\int_{S+\Gamma}H(t)\ln(t-z)dt+g, \tag{6}$$

$$\psi(z) = \int \Psi(z)dz = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty + 2i\sigma_{12}^\infty)z + \frac{1}{2\pi i}\int_{S+\Gamma}\left\{\left[\overline{H(t)} - \overline{q(t)}\right]e^{-2i\alpha(t)}\ln(t-z) - \frac{\bar{t}H(t)}{t-z}\right\}dt + \gamma', \tag{7}$$

$$H(t) = \frac{1}{\kappa+1}[q(t) + w(t)e^{-i\alpha(t)}] \qquad for\ t = y_1 + iy_2 \in S + \Gamma, \tag{8}$$

Before we continue to consider the boundary values of our complex functions, it is useful to note from (8) that *q(t)* can be expressed in terms of *H(t)* and *w(t)* as

$$q(t) = (k+1)H(t) - w(t)e^{-ia(t)} \qquad for\ t = y_1 + iy_2 \in S + \Gamma. \tag{9}$$

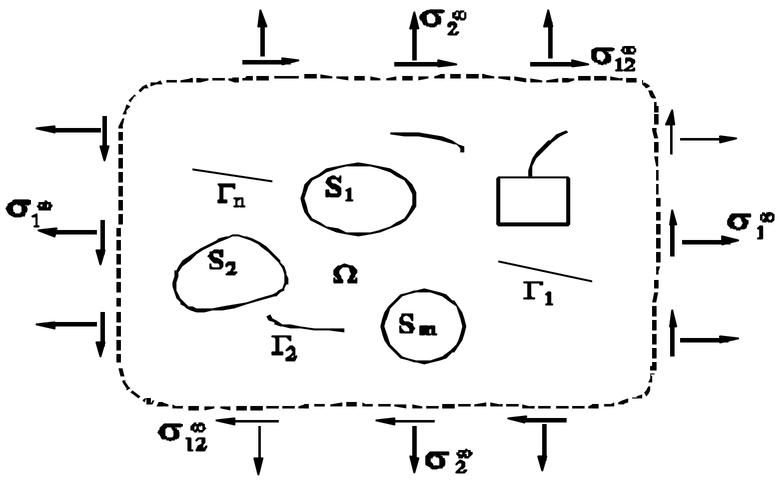

Figure 1: A sketch for an infinite elastic body containing *n* cracks $_j$ (j = 1,...,*n*) and *m* holes S_i (i=1,..., m) subjected to far field stresses s_1^∞, s_2^∞ and s_{12}^∞.

In obtaining the above formula, we have let the outer boundary tends to infinity and the components of stress at infinity are given as s_1^∞, s_2^∞ and s_{12}^∞. This formulation bears a close resemblance with Muskhelishvili's formalism (1975). In these formulas, S denotes the union of the holes $S_1, S_2, \ldots, S_m$, $C = 2Ge_\infty/(k+1)$ with e_∞ being the rotation at infinity, and the outer boundary S_0, and G the union of the cracks G_1, G_2, ..., G_n. The shear modulus and Poisson's ratio are denoted by G and respectively. The plane parameter equals 3 4 for plane strain or (3)/(1+) for plane stress. The angle between the tangent at *t* on S+ and the global coordinate axis ox_1 is denoted by (*t*). And $_{ij}$ and u_i (*i,j* =1,2) are the components of stress and displacement in the Cartesian coordinate system ox_1x_2, respectively. $_n$ and $_{ns}$ respectively are the normal and shear stresses on the boundary. The superscripts "+" and " " denote the upper and lower crack faces respectively. The complex integration constants and relate only to rigid displacements.

It is obvious that the only unknown function in the boundary integral formulation for (*t*), (*t*), (*t*) and (*t*) is *H*(*t*). Therefore, only one variable is needed in this complex formulation and this is one of the main advantage of using the present complex formulation.

MIXED BOUNDARY INTEGRAL EQUATIONS

The stresses and displacements shown above satisfy automatically the equilibrium equations and the displacement-strain relations. In addition, they must also satisfy the boundary conditions, which will lead to the boundary integral equations for the unknown boundary complex function $H(t)$. For infinite plane elastic bodies containing cracks and holes shown in Figure 1, Wang and Chau [1] obtained the following BIEs for mixed BVPs

$$\int_{S+\Gamma}\left\{\frac{H(t)}{t-t_0}-\frac{\overline{H(t)}}{\bar{t}-\bar{t}_0}e^{-2i\alpha(t)}+e^{-2i\alpha(t_0)}\left[\frac{H(t)}{\bar{t}-\bar{t}_0}+\frac{t-t_0}{(\bar{t}-\bar{t}_0)^2}\overline{H(t)}e^{-2i\alpha(t)}\right]\right\}dt$$

$$-(\kappa+1)e^{-2i\alpha(t_0)}\int_{S_u+\Gamma_u}\frac{H(t)dt}{\bar{t}-\bar{t}_0}+e^{-2i\alpha(t_0)}\int_{S_u+\Gamma_u}\frac{w(t)e^{-i\alpha(t)}dt}{\bar{t}-\bar{t}_0}-e^{-2i\alpha(t_0)}\int_{S_t+\Gamma_t}\frac{q(t)dt}{\bar{t}-\bar{t}_0}=\pi i[f(t_0)-g_1(t_0)]$$

$$(t_0=x_{01}+ix_{02}\in S_t+\Gamma_t); \tag{10}$$

$$\int_{S+\Gamma}\left\{\frac{\kappa H(t)}{t-t_0}+\frac{\overline{H(t)}}{\bar{t}-\bar{t}_0}e^{-2i\alpha(t)}+e^{-2i\alpha(t_0)}\left[\frac{\kappa H(t)}{\bar{t}-\bar{t}_0}-\frac{t-t_0}{(\bar{t}-\bar{t}_0)^2}\overline{H(t)}e^{-2i\alpha(t)}\right]\right\}dt$$

$$-(\kappa+1)e^{-2i\alpha(t_0)}\int_{S_t+\Gamma_t}\frac{H(t)dt}{\bar{t}-\bar{t}_0}-e^{-2i\alpha(t_0)}\int_{S_u+\Gamma_u}\frac{w(t)e^{-i\alpha(t)}dt}{\bar{t}-\bar{t}_0}+e^{-2i\alpha(t_0)}\int_{S_t+\Gamma_t}\frac{q(t)dt}{\bar{t}-\bar{t}_0}=\pi i[v(t_0)-g_2(t_0)]$$

,

$$(t_0=x_{01}+ix_{02}\in S_u+\Gamma_u); \tag{11}$$

where

$$g_1(t_0)=\sigma_1^{\infty}+\sigma_2^{\infty}+(\sigma_2^{\infty}-\sigma_1^{\infty}-2i\sigma_{12}^{\infty})e^{-2i\alpha(t_0)}, \tag{12}$$

$$g_2(t_0)=\frac{\kappa-1}{2}(\sigma_1^{\infty}+\sigma_2^{\infty})-(\sigma_2^{\infty}-\sigma_1^{\infty}-2i\sigma_{12}^{\infty})e^{-2i\alpha(t_0)}+2(\kappa+1)Ci; \tag{13}$$

and $f(t_0)$ and $v(t_0)$ are given BY Wang and Chau [1]. In deriving these BIEs, we have used the Plemelj formulas (Muskhelishvili [13-14]; England [25]) and the following formulas of $h_2(z)$ (Wang and Chau [1]):

$$\begin{aligned} h_2^{+}(t_0)&=\frac{1}{2}H(t_0)e^{-2i\alpha(t_0)}+\frac{1}{2\pi i}\int_{S+\Gamma}\frac{\bar{t}-\bar{t}_0}{(t-t_0)^2}H(t)dt,\\ h_2^{-}(t_0)&=-\frac{1}{2}H(t_0)e^{-2i\alpha(t_0)}+\frac{1}{2\pi i}\int_{S+\Gamma}\frac{\bar{t}-\bar{t}_0}{(t-t_0)^2}H(t)dt; \end{aligned} \qquad (t_0\in S+\Gamma) \tag{14}$$

COMPATIBILITY CONDITION

In the case of multi-connected region, the unknown boundary function $H(t)$ for infinite bodies must satisfy the following compatibility conditions (Chau and Wang [20]):

$$\int_{S_k}H(t)dt=\frac{1}{\kappa+1}\int_{S_k}q(t)dt \quad \text{for every hole } S_k \text{ (k=1,2,...,m)}, \tag{15}$$

$$\int_{\Gamma_j}H(t)dt=\frac{1}{\kappa+1}\int_{\Gamma_j}q(t)dt \quad \text{for every embedded crack } \Gamma_j \text{ (j=1,2,...,n)}. \tag{16}$$

The mixed BIEs (10-11) must be solved in conjunction with the compatibility conditions (15-16), either analytically or numerically by using BEM similar to those discussed by Wang and Chau [21]. Once the boundary unknown $H(t)$ is obtained, the complex functions (z), (z), (z) and (z) can be determined. Subsequently, the stress and displacement components can be calculated.

NUMERICAL IMPLEMENTATION

The boundaries of any elastic body containing cracks and holes, either traction or displacement boundary, are discretized into a number of linear elements. Each element L_e is then mapped onto the interval 1 1. Linear shape functions are adopted for both complex variable t and the complex boundary function $H(t)$ on the non-singular crack elements. For crack tips, a square-root singularity is assumed (Wang and Chau [1]). In the case that the complex boundary function $H(t)$ on the hole's boundary, an additional constant is introduced for each hole such that the compatibility can be satisfied.

Once the solutions for the nodal unknowns are obtained by numerical calculations, the stress intensity factors can be determined from the following equations (Wang and Chau [21]):

$$K_I(a_j) - iK_{II}(a_j) = -\lim_{t \to a} \sqrt{2\pi|t-a_j|} \cdot iH(t), \tag{17}$$

$$K_I(b_j) - iK_{II}(b_j) = \lim_{t \to b} \sqrt{2\pi|t-b_j|} \cdot iH(t); \qquad (j=1,2,...,n) \tag{18}$$

where a_j and b_j are two tips of the crack $_j$.

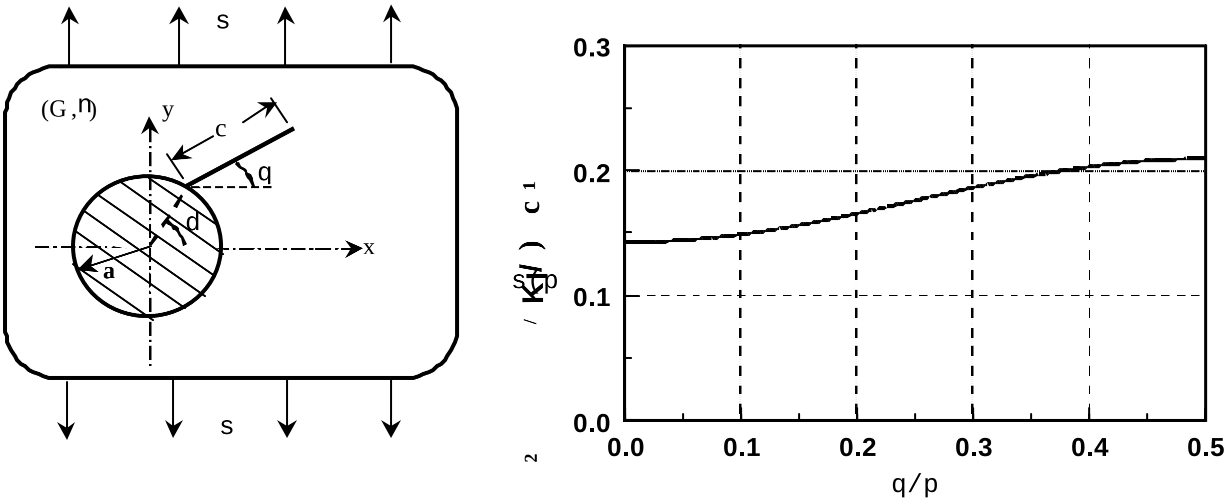

Figure 2: A crack of length c emanating from the interface of a circular rigid inclusion and an elastic matrix at the point measuring from the x-axis, and inclining at under tension . The mode I stress intensity factor is given for the case of =0.25, = and c=0.1a (after Wang and Chau [21])

NUMERICAL RESULTS AND CONCLUSION

Consider the case that a crack is emanating from a rigid inclusion (Fig. 2), the mode I crack tip stress intensity factor has been calculated by Wang and Chau [1]. Figure 2 plots the normalized mode I stress intensity factors for = and c=0.1a. Wang and Chau [1] also show that the present results are comparable to Gharpuray et al. [3] when the inclusion is relatively rigid. Thus, the validity of the present BEM is demonstrated.

In this paper, the new BEM formulation by Wang and Chau [1] is presented. Although only the results for radial crack is presented here, the present BEM has also been applied to consider the interaction between a rigid circular inclusion and a crack, either an edge crack emanating from the interface or an internal crack in the elastic matrix (Wang and Chau [1]). For the case of rigid inclusion, Wang and Chau [1] has shown that our solutions are comparable to those by Erdogan et al. [4], Erdogan and Gupta [2], Isida and Noguchi [12], and Gharpuray et al. [3].

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