Focussed on Multiaxial Fatigue and Fracture

A comparison of methods for calculating notch tip strains and stresses under multiaxial loading

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ABSTRACT. Selected methods for calculating notch tip strains and stresses in elastic–plastic isotropic bodies subjected to multiaxial monotonic loading were compared. The methods use sets of equations where hypothetical notch tip elastic strains and stresses obtained from FEM calculations serve as an input. The comparison was performed within two separate groups of methods: the first group consists of the methods intended for cases of multiaxial proportional loading and the second group deals with multiaxial non-proportional loading. Originally, the precision of the methods was validated by comparison with results obtained from elastic–plastic FEM analyses. Since computer performance at the time was lower than nowadays, verification of the proposed methods on FEM models with a finer mesh was needed. Such verification was carried out and is presented in this paper. The effect of various formulations of material stress–strain curve was also evaluated.

KEYWORDS. Notch stresses; Multiaxial loading; FEM; Neuber's rule; ESED rule.

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INTRODUCTION

There are several ways to obtain notch tip stresses and strains: numerical simulation, experiment, and estimation methods. Since the first two approaches are time consuming, it is desirable to use estimation methods that provide results based on elastic FE simulation only. Such methods have been developed since the last century and can work with different types of loading, including cyclic loading [1], [2]. The essence of those methods is in using results from elastic FE analyses and converting these results into elastic–plastic estimates using suitable sets of equations.

In this paper, two sets of methods were selected for comparisons. The first set consisted of the methods intended for multiaxial proportional monotonic loading. The selected methods were Moftakhar's method [3], Reinhardt's method [4], and Hoffmann–Seeger's method [5]. The second set consisted of the methods intended for multiaxial non-proportional monotonic loading. The selected methods were Singh's method [6] and Buczynski's method [7].

SELECTED METHODS

Due to the plane stress state at the notch tip, one component of principal stresses has a zero value and therefore, in case of principal components, there are five unknowns, for which five equations are needed. These unknowns are maximum principal strain ε_1, middle principal strain ε_2, minimum principal strain ε_3, and two of principal stresses σ_1, σ_2, σ_3 depending on the type of loading.

All of the selected methods, with the exception of Hoffmann–Seeger's method, were defined in two different variants. The first estimate is obtained when Neuber's rule is used for compiling the set of equations. The second estimate is obtained when the ESED rule [3], [6] is used instead. Hoffmann–Seeger's method uses only Neuber's rule.

Both Neuber's rule and the ESED rule state that strain energy density at the notch tip of a body with elastic–plastic behaviour is equal to strain energy density at the notch tip of a body with ideal elastic behaviour. The difference between these rules is that Neuber's rule also takes in account complementary strain energy density [3].

As the first equation for prediction according to Moftakhar's method, the extensions either of Neuber's rule or the ESED rule are used. These extensions were developed for multiaxial loading and are represented by a sum of products of the individual stress and strain components.

The next three equations for Moftakhar's method are provided by Hencky's equations that relate strains, stresses, and a material model represented by the cyclic stress–strain curve.

The last equation is provided by the energy distribution assumption which states that fractional contribution of the largest principal notch tip stresses and strains to the total notch tip strain energy density [3] is equal for a body with elastic–plastic behaviour and a for a body with elastic behaviour.

Besides the five unknown parameters mentioned above, there is another unknown which is a part of Hencky's equations. This parameter is the plastic component of equivalent deformation. Moftakhar in [3] calculates it from bilinear approximation of cyclic stress–strain curve. Another way to calculate this parameter is by using the Ramberg–Osgood expression.

Reinhardt's method [4] represents a modification of Moftakhar's method. The motivation for the modification was the fact that, since the original method forms a set of nonlinear algebraic equations, several solutions for that set exist in general [4]. Therefore, Reinhardt et al. presented a new algorithm for calculating notch tip stresses and strains based on the equations derived from the original Moftakhar's set.

In Hoffmann–Seeger's method [5], another generalized form of Neuber's rule is used for calculating actual equivalent stress. The method also uses Hencky's expression, expression for von Mises stress, and an assumption about the ratio of surface strains to calculate the unknown components of stress and strain tensors.

The selected methods designed for non-proportional loading work with general components of stresses and strains, and therefore there are seven unknowns (ε_{11}, ε_{22}, ε_{23}, ε_{33}, σ_{22}, σ_{23}, σ_{33}), for which seven equations are needed. Since the ratio of deviatoric stresses in case of non-proportional loading is unlikely to be a constant value, the final stress–strain state is dependent on the applied loading path [6]. Therefore, the equations used for predictions should be in an incremental form.

The set of equations for Singh's method [6] consists of a) either generalized Neuber's rule or the ESED rule, b) the Prandtl–Reuss relations, which reflect elastic–plastic material behaviour and provide four equations, and c) two equations provided by energy ration assumptions.

The first three equations for Buczynski's method [7] are obtained from either generalized Neuber's or the ESED rule, expressed as an equality of symmetric strain energy density tensors. The remaining four equations are provided by the Prandtl–Reuss relation.

VERIFICATION PROCESS

The verification process was as follows: at first the selected methods were implemented in two programs written in MATLAB, one for the case of proportional loading, the other one for non-proportional loading.

In the next step, in order to verify whether individual methods are implemented correctly, the data from elastic simulations referred originally by the authors, were derived from the articles.

After obtaining the original inputs, they were imported into the MATLAB programs. At the beginning of the estimation process, input vectors were divided by preselected increments. In case of methods designed for non-proportional loading, the end of the first increment was the yielding state. The yielding state was found by linear interpolation using von Mises stress (Eq. 1) for comparison with yield strength

$$\sigma_{eq}^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \qquad (1)$$

where σ_{eq} is equivalent stress.
In all cases, estimations started from the yielding point. In case of proportional loading it was only natural since the modification of inputs was performed so that the first non-zero input values corresponded to the yielding state. In non-proportional load cases, it was checked whether the value of equivalent stress was higher than yield strength. If that condition was not met, the solutions for the current step were equal to the elastic solution.
The solutions to the sets of equations provided by each method were obtained using MATLAB function *fsolve*.
The results of estimations were compared with the elastic–plastic FE solutions presented in the original articles. The only exception was Reinhardt's method, the results of which were compared with Moftakhar's method.
In case of non-proportional loading, besides the multilinear approximation of cyclic stress–strain curve, the material model described by the Ramberg–Osgood expression was used to compare the quality of estimates.
After the verification of the implementation of the methods, the finite element analyses were carried out. The aim of those analyses was to provide inputs that were obtained on FE models with a mesh denser than used to be common, when the evaluated methods were first presented. To get a model with a representative fine mesh, sensitivity analyses were carried out on FE models with an elastic material model and different mesh densities. The appropriate model was then the model for which changes in values of equivalent stress were insignificant (less than 1% for cases of proportional loading, and less than 4% for cases of non-proportional loading).
For proportional loading, simulations on two types of specimens were performed. The first one was a solid bar with a circumferential groove loaded simultaneously with tension and torsion. Another specimen was a hollow tube with a circumferential groove loaded by internal pressure and tension. Both types of specimens and loads corresponded to those presented in [3]. For non-proportional loading, a round bar with a circumferential groove loaded by torsion in the first step and by increasing tension and constant torsion in the second step, was used. This type of model is presented in [6].
After results from FE simulations were obtained, they were used as inputs for estimations in order to assess and compare prediction quality of selected methods.

RESULTS OF VERIFICATION OF METHODS IMPLEMENTATION

The same results as presented in [3] were obtained for Moftakhar's method using 4 increments in elastic–plastic region and bilinear approximation as the material model. Same settings were used for Reinhardt's method and it led to the same results as in case of Moftakhar's method. The implementation of Hoffmann–Seeger's method was verified using the results from [5].
The same results as in [6] were obtained for Singh's method using 28 increments per loading step and 6 linear approximation of the cyclic stress–strain curve. The attempt to get the same results as in [7] for Buczynski's method was not successful. The *fsolve* function failed to solve the set of equations defined by the method. Numerical instability was observed for both definitions of the material model, with the Ramberg–Osgood expression or with the linear approximation. It was also observed that changing the settings of *TolFun* parameter, which defines termination tolerance on the *fsolve* function value [8], affects this behaviour. The smallest instability is observed in the case of 6 piecewise linear material model when estimations start from plasticized state with settings of *TolFun* to 1e-5.

COMPARISON WITH USE OF MODERN FE MODELS

Proportional loading

In Fig. 1, the estimates of stress and strain components are presented.
The stress estimates are very similar to those presented in [3]. There is a slight difference in minimum principal stress component for the bar specimen at the end of load sequence, where the prediction is most accurate as the elastic–plastic result is bounded by two Moftakhar's or two Reinhardt's estimates. There is also a slight difference in case of the hollow tube, where the estimate according to Moftakhar's method with the ESED rule is closer to the elastic–plastic results from the FE simulation.

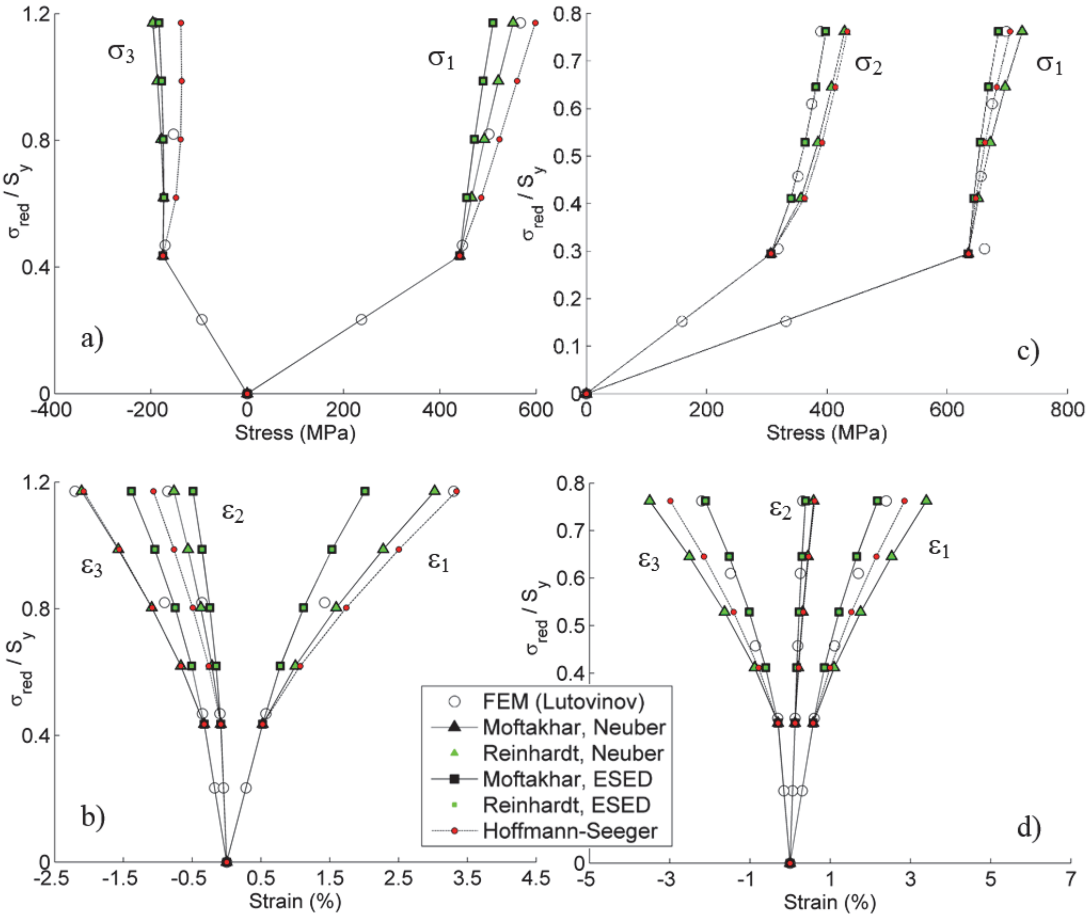

Figure 1: Results of estimations carried out on the round specimen (a, b) and the hollow tube (c, d) [3]. σ_{red} is equivalent nominal net stress and S_y is yield strength.

For maximum principal stresses, Hoffmann–Seeger's method seemed to be a better upper bound limit than Moftakhar's method with Neuber's rule.

As in case of stresses obtained on the hollow tube, elastic–plastic strains from the FEM simulation and from the prediction by Moftakhar's method with the ESED rule are closer than presented in [3]. Maximum principal strain could be well predicted with using Hoffman–Seeger's method as the upper limit and Moftakhar's method with the ESED rule as the lower limit. Minimum and middle principal strains could be obtained by using Hoffman–Seeger's method. All methods however are slightly non-conservative in case of minimum strain at round bar.

Non-proportional loading

Estimates of stress components σ_{22} and σ_{33} according to Singh's method are more precise than estimates presented originally in [6].

In case of shear stress component, once the yield stress is exceeded, all four estimations provide non-conservative results. Estimates according to Buczynski's and Singh's methods are identical in this region. In the end of the load sequence, Buczynski's method with the ESED rule provides the most accurate estimate.

Shear strain component calculated in the elastic–plastic FEM model with the fine mesh result in substantially higher values than those referred to in [6]. It is possible to use Singh's methods as the upper and the lower limit. The use of average values of those two methods would lead to a slightly non-conservative solution for the component. Other components are predicted quite well.

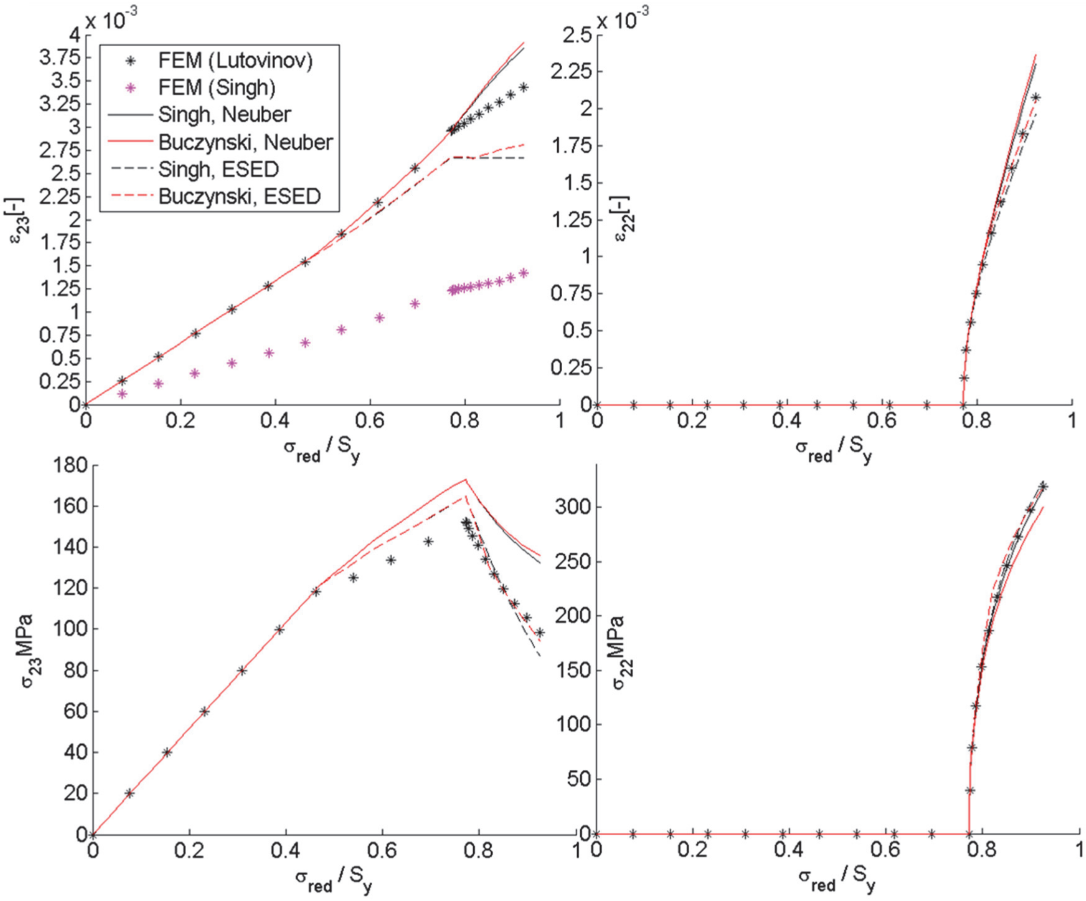

Figure 2: Results of estimations carried out on the bar specimen loaded non-proportionally with subsequent tension and torsion [6]. Estimations were performed with the use of 6-linear approximation of cyclic stress–strain curve.

Using the Ramberg–Osgood material model

The predictions using the Ramberg–Osgood material model led to worse estimation quality in general.

In case of stresses, all estimates are shifted to the conservative region, while strains are shifted to the non-conservative region. The estimate of the shear stress component by Buczynski's method with the ESED rule changed more than all other predictions and led to a distinctly non-conservative result.

CONCLUSIONS

- Quality of estimates of all methods is substantially dependent on used inputs. Therefore, further investigation is needed in order to determine the extent of this dependence. For the same reason, it is also desirable to apply the methods to other types of specimens and to experimental data.

Proportional loading

- For calculating maximum principal stresses and strains, Hoffmann–Seeger's method should be used as the upper limit. As the lower limit, Moftakhar's method with the ESED rule is the best option from those considered.
- For calculating middle and minimum principal stress and strain components, both Moftakhar's estimations can be successfully used as the lower and the upper limit.

Non-proportional loading

- Due to the high dependence of the estimates according to Buczynski's method on settings of the solver function and based on the observation that predictions according to the Buczynski's method and Singh's method are very similar, it is proposed to use Singh's method.

- The use of the multilinear approximation of the cyclic stress–strain curve for estimations led to better results than the use of the Ramberg–Osgood description of the cyclic stress–strain curve.

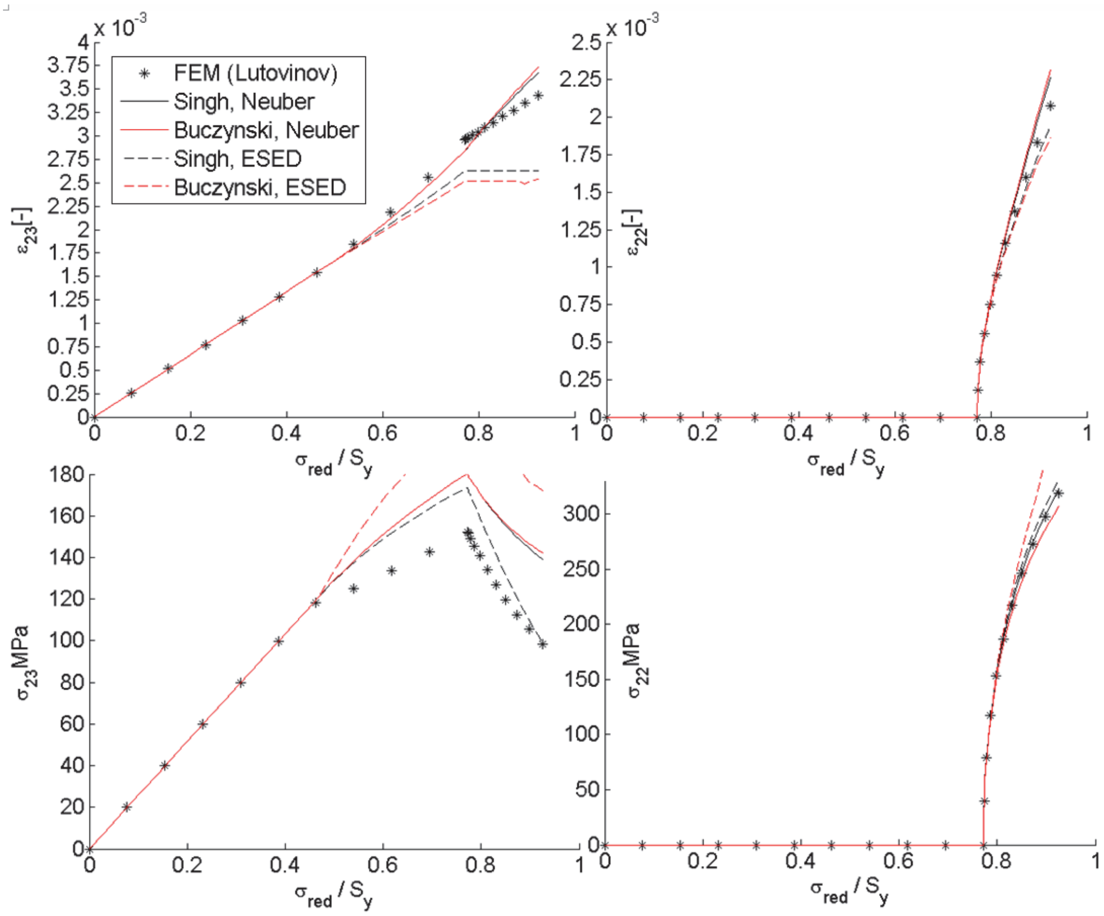

Figure 3: Results of estimations carried out on bar specimen loaded non-proportionally with subsequent tension and torsion [6]. Estimations were performed with the use of cyclic stress–strain curve described by the Ramberg–Osgood expression.

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