SINGULAR HARMONIC PROBLEMS AT MULTI-MATERIAL WEDGES: MATHEMATICAL ANALOGIES BETWEEN ELASTICITY, DIFFUSION AND ELECTROMAGNETISM

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ABSTRACT

Multi-material wedges are frequently observed in composite materials. They consist of two or more sectors of dissimilar materials joined together, whose interfaces converge to the same vertex. Due to the mismatch in the material properties, such as Young's modulus, thermal conductivity, dielectric permittivity, or magnetic permeability, these geometrical configurations may lead to singular fields at the junction vertex. In this paper, focusing the attention on singular harmonic problems, the mathematical analogies intercurring between antiplane shear problem in elasticity due to Mode III loading or torsion, the steady-state heat transfer problem, and the diffraction of waves in electromagnetism are presented. The proposed unified mathematical formulation is particularly convenient for the identification of common types of singularities (power-law or logarithmic type), for the use of a standardized method for solving the nonlinear eigenvalue problems, and for the determination of common geometrical and material configurations permitting to relieve or remove the singularities.

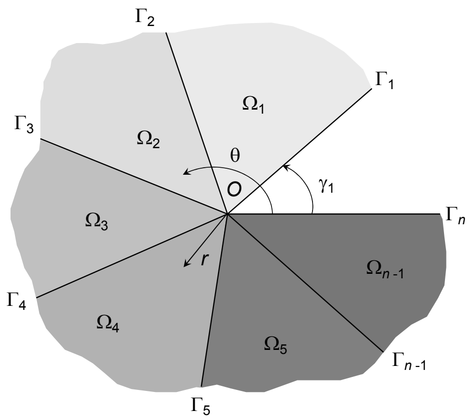
KEYWORDS

Singularities, multi-material wedges, elasticity, diffusion, electromagnetism.

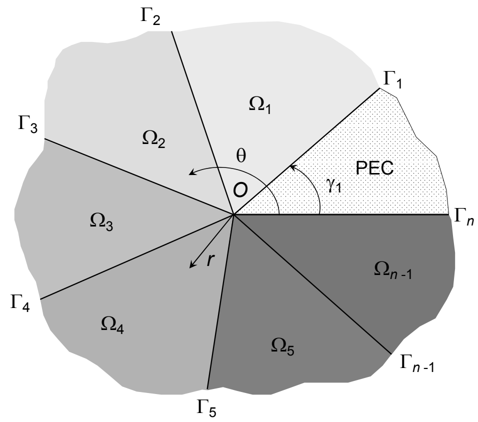
INTRODUCTION

Singular stress states occur in several boundary value problems of linear elasticity where different materials are present (see [1-3] for a wide overview). In this context, the problems of multi-material wedges or junctions have received a great attention from the scientific community, since they are commonly observed in composite materials. In linear elasticity, most of the research efforts have been directed to the characterization of stress-singularities for in-plane loading, where the problem is governed by a biharmonic equation. The out-of-plane loading, also referred to as antiplane shear problem, is governed by a simpler harmonic equation. Stress-singularities due to antiplane loading were firstly addressed by Rao [4] in 1971. Afterwards, Fenner [5] examined the Mode III loading problem of a crack meeting a bi-material interface using the eigenfunction expansion method proposed by Williams [6]. More recently, Ma and Hour [7] analyzed bi-material wedges using the Mellin transform technique and Pageau et al. [8] investigated the singular stress field of bonded and debonded tri-material junctions according to the eigenfunction expansion method.

In 1980, Sinclair [9] discovered the mathematical analogy intercurring between the steady-state heat transfer and the antiplane loading of composite regions (see also [3]). Very recently, Paggi et al. [10] have established the analogy between elasticity and electromagnetism. In the solution of diffraction problems, in fact, Bouwkamp [11] and Meixner [12] found that the electromagnetic field vectors may become infinite at the sharp edges of a diffracting obstacle. For in-plane problems, a mathematical analogy between elasticity and fluid dynamics also exists, see [3,13,14] for more details. In this paper, the analogies for singular harmonic problems are briefly reviewed and a unified mathematical formulation is presented. More specifically, the eigenfunction expansion method is adopted, which has been proven



(a) Multi-material wedge.



(b) Multi-material wedge with a PEC material.

Figure 1: Multi-material wedges in elasticity, diffusion and electromagnetism.

in [3] to be mathematically equivalent to the Muskhelishvili complex function representation and to the Mellin transform technique for the characterization of elastic singularities at multimaterial junctions. As a main outcome, the order of the stress-singularities of various geometrical and mechanical configurations already determined in the literature can be adopted for the analogous diffusion and electromagnetic problems, without the need of performing new calculations.

STRESS SINGULARITIES IN ANTIPLANE ELASTICITY

The geometry of a plane elastostatic problem consisting of $n-1$ dissimilar isotropic, homogeneous sectors of arbitrary angles perfectly bonded along their interfaces converging to the same vertex O is shown in Fig. 1(a). Each of the material regions is denoted by Ω_i with $i = 1, \ldots, n-1$, and it is comprised between the interfaces Γ_i and Γ_{i+1}.
Antiplane shear (Mode III) due to out-of-plane loading on composite wedges can lead to stresses that can be unbounded at the junction vertex O. When out-of-plane deformations only exist, the following displacements in cylindrical coordinates can be considered with the origin at the vertex O:

$$u_r = 0, \quad u_\theta = 0, \quad u_z = u_z(r, \theta), \tag{1}$$

where u_z is the out-of-plane displacement, which depends on r and θ. For such a system of displacements, the strain field components become

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = 0, \tag{2a}$$

$$\gamma_{rz} = \frac{\partial u_z}{\partial r}, \quad \gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta}. \tag{2b}$$

From the application of the Hooke's law, the stress field components are given by:

$$\sigma_r^i = \sigma_\theta^i = \sigma_z^i = \tau_{r\theta}^i = 0, \tag{3a}$$

$$\tau_{rz}^i = G_i \gamma_{rz}^i = G_i \frac{\partial u_z^i}{\partial r}, \quad \tau_{\theta z} = G_i \gamma_{\theta z}^i = \frac{G_i}{r}\frac{\partial u_z^i}{\partial \theta}, \tag{3b}$$

where G_i is the shear modulus of the i-th material region. The equilibrium equations in absence of body forces reduce to a single relationship between the tangential stresses:

$$\frac{\partial \tau_{rz}^i}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}^i}{\partial \theta} + \frac{1}{r}\tau_{rz}^i = 0, \quad \forall (r, \theta) \in \Omega_i. \tag{4}$$

Introducing Eqs. (3) into Eq. (4), the harmonic condition upon u_z is derived:

$$\frac{\partial^2 u_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial u_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z^i}{\partial \theta^2} = \nabla^2 u_z^i = 0,\ \forall (r,\theta) \in \Omega_i. \tag{5}$$

In the framework of the eigenfunction expansion method [6], the following separable variable form for the longitudinal displacement u_z^i can be adopted ($\forall (r,\theta) \in \Omega_i$):

$$u_z^i(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}\ (\theta, \lambda_j), \tag{6}$$

where λ_j are the eigenvalues of the problem and $f_{i,j}$ the eigenfunctions. The summation with respect to the subscript j is introduced in Eq. (6), since it is possible to have more than one eigenvalue and the Superposition Principle can be applied.

Introducing Eq. (6) into Eq. (5), we find the following relationship holding for each eigenvalue λ_j:

$$r^{\lambda_j - 2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2} + \lambda_j^2 f_{i,j}\right) = 0. \tag{7}$$

Hence, the coefficients of the term in r^{λ_j-2} must vanish, implying that the eigenfunctions $f_{i,j}$ are a linear combination of trigonometric functions:

$$f_{i,j}(\theta,\lambda_j) = A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta). \tag{8}$$

If we introduce the series expansion (6) into Eqs. (3), the longitudinal displacement and the tangential stresses can be expressed in terms of the eigenfunction and its first derivative:

$$u_z^i = r^{\lambda_j} f_{i,j} = r^{\lambda_j}\left[A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta)\right], \tag{9a}$$

$$\tau_{rz}^i = G_i\lambda_j r^{\lambda_j-1} f_{i,j} = G_i\lambda_j r^{\lambda_j-1}\left[A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta)\right], \tag{9b}$$

$$\tau_{\theta z}^i = G_i r^{\lambda_j-1} f_{i,j}' = G_i\lambda_j r^{\lambda_j-1}\left[A_{i,j}\cos(\lambda_j\theta) - B_{i,j}\sin(\lambda_j\theta)\right]. \tag{9c}$$

The determination of the power of the stress-singularity, $\lambda_j - 1$, can be performed by imposing the boundary conditions (BCs) along the edges Γ_1 and Γ_n and at the bi-material interfaces Γ_i, with $i = 2,\ldots,n-1$. Along the edges Γ_1 and Γ_n, defined by the angles γ_1 and γ_n, we consider two possibilities: one corresponding to unrestrained stress-free edges

$$\tau_{\theta z}^i(r,\gamma_1) = 0, \quad \tau_{\theta z}^i(r,\gamma_n) = 0, \tag{10}$$

and the other for fully restrained (clamped) edges

$$u_z^i(r,\gamma_1) = 0, \quad u_z^i(r,\gamma_n) = 0. \tag{11}$$

At the interfaces, the following continuity conditions of displacements and stresses have to be imposed ($i = 1,\ldots,n-2$):

$$u_z^i(r,\gamma_{i+1}) = u_z^{i+1}(r,\gamma_{i+1}), \quad \tau_{\theta z}^i(r,\gamma_{i+1}) = \tau_{\theta z}^{i+1}(r,\gamma_{i+1}). \tag{12}$$

In this way, a set of $2n-2$ homogeneous equations in the $2n-1$ unknowns $A_{i,j}$, $B_{i,j}$, and λ_j can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{v} = \mathbf{0}, \tag{13}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue, and $\mathbf{v}$ represents the vector that collects the unknowns $A_{i,j}$ and $B_{i,j}$. More specifically, the coefficient matrix entering Eq. (13) is characterized by a sparse structure:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{N}^1_{\gamma_1} & & & & & & & \\ \mathbf{M}^1_{\gamma_2} & -\mathbf{M}^2_{\gamma_2} & & & & & & \\ & \mathbf{M}^2_{\gamma_3} & -\mathbf{M}^3_{\gamma_3} & & & & & \\ & & \cdots & \cdots & & & & \\ & & & \mathbf{M}^{i-1}_{\gamma_i} & -\mathbf{M}^i_{\gamma_i} & & & \\ & & & & \cdots & \cdots & & \\ & & & & & \mathbf{M}^{n-2}_{\gamma_{n-1}} & -\mathbf{M}^{n-1}_{\gamma_{n-1}} \\ & & & & & & \mathbf{N}^{\mathrm{n}-1}_{\gamma_\mathrm{n}} \end{bmatrix} \tag{14}$$

where the non null elementary matrix $\mathbf{M}^i_\theta$ related to the interface BCs is given by:

$$\mathbf{M}^i_\theta = \begin{bmatrix} \sin(\lambda_j\theta) & \cos(\lambda_j\theta) \\ G_i\cos(\lambda_j\theta) & -G_i\sin(\lambda_j\theta) \end{bmatrix} \tag{15}$$

and the components of the vector $\mathbf{v}$ are:

$$\mathbf{v} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^i, \ldots, \mathbf{v}^{n-2}, \mathbf{v}^{n-1}\}, \tag{16}$$

with $\mathbf{v}^i = \{A_{i,j}, B_{i,j}\}^T$. The two remaining terms $\mathbf{N}^i_\theta$ depend on the BCs along the edges Γ_1 and Γ_n. For stress-free edges we have:

$$\mathbf{N}^i_\theta = \{\cos(\lambda_j\theta), -\sin(\lambda_j\theta)\}, \tag{17}$$

whereas for clamped edges it is given by

$$\mathbf{N}^i_\theta = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}. \tag{18}$$

A nontrivial solution of the equation system (13) exists if and only if the determinant of the coefficient matrix vanishes. This condition yields an eigenequation which has to be solved for the eigenvalues λ_j that, in the most general case, do depend on the elastic properties of the materials.

HEAT FLUX SINGULARITIES IN DIFFUSION PROBLEMS

The analogy between steady-state heat transfer and antiplane shear in composite regions was discovered by Sinclair [9] in 1980. In both problems, the field equations for the longitudinal displacement, u^i_z, and for the temperature, T^i, are harmonic. As a result, the following correspondences between these two problems can be settled down:

$$\begin{aligned} \nabla^2 T^i = 0 \quad &\Leftrightarrow \quad \nabla^2 u^i_z = 0, \\ q^i_r = -k_i\frac{\partial T^i}{\partial r} \quad &\Leftrightarrow \quad \tau^i_{rz} = G_i\frac{\partial u^i_z}{\partial r}, \\ q^i_\theta = \frac{k_i}{r}\frac{\partial T^i}{\partial \theta} \quad &\Leftrightarrow \quad \tau^i_{\theta z} = \frac{G_i}{r}\frac{\partial u^i_z}{\partial \theta}, \end{aligned} \tag{19}$$

where q^i_r and q^i_θ are, respectively, the heat flux in the radial and circumferential directions and k_i is the thermal conductivity in the i-th material region. Therefore, the analogy is straightforward: the temperature field is analogous to the out-of-plane displacement field, whereas the heat flux components are the analogous counterparts of the the stress field components, diverging to infinity as $r \to 0$.

As far as the BCs are concerned, the free-edge conditions (10) correspond to insulated edges in diffusion problems, provided that the elastic variables are replaced by the steady-state heat transfer variables according to (19). Similarly, the clamped BCs (11) in elasticity correspond to zero temperature prescribed along the edges. Finally, the continuity of the longitudinal displacement u_z and of the tangential stress $\tau_{\theta z}$ in Eq. (12) at the interfaces corresponds to the continuity of temperature, T, and heat-flux, q_θ. The eigenvalue problem for the diffusion problem has therefore the same coefficient matrix as in Eq. (13).

SINGULARITIES IN THE ELECTRO-MAGNETIC FIELDS

Let us consider the multi-material wedge shown in Fig. 1(b). Each material is isotropic and has a dielectric permittivity ϵ_i and a magnetic permeability μ_i. We also admit the presence of a perfect electric conductor (PEC) in the region 1 defined by the interfaces Γ_1 and Γ_n. For periodic fields with circular frequency ω, the Maxwell's equations for each homogeneous angular domain read [12]:

$$\mathrm{j}\omega\epsilon_i\mathbf{E}^i = \nabla\times\mathbf{H}^i, \quad -\mathrm{j}\omega\mu_i\mathbf{H}^i = \nabla\times\mathbf{E}^i, \tag{20}$$

where $\mathbf{E}^i$ and $\mathbf{H}^i$ are, respectively, the electric and magnetic fields, and the symbol j stands for the imaginary unit.

In cylindrical coordinates r, θ, z, with the z axis perpendicular to the plane of the wedge, and considering electromagnetic fields independent of z, the Maxwell's equations reduce to the following conditions upon the components of the electric and magnetic fields:

$$\mathrm{j}\omega\epsilon_i E_r^i = \frac{1}{r}\frac{\partial H_z^i}{\partial\theta}, \tag{21a}$$

$$\mathrm{j}\omega\epsilon_i E_\theta^i = -\frac{\partial H_z^i}{\partial r}, \tag{21b}$$

$$\mathrm{j}\omega\epsilon_i E_z^i = \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta^i) - \frac{1}{r}\frac{\partial H_r^i}{\partial\theta}, \tag{21c}$$

$$-\mathrm{j}\omega\mu_i H_r^i = \frac{1}{r}\frac{\partial E_z^i}{\partial\theta}, \tag{21d}$$

$$-\mathrm{j}\omega\mu_i H_\theta^i = -\frac{\partial E_z^i}{\partial r}, \tag{21e}$$

$$-\mathrm{j}\omega\mu_i H_z^i = \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta^i) - \frac{1}{r}\frac{\partial E_r^i}{\partial\theta}. \tag{21f}$$

It is easy to verify that the E_z^i and H_z^i components satisfy the Helmholtz equation [15]:

$$\frac{\partial^2 E_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial E_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z^i}{\partial\theta^2} + k_i^2 E_z^i = \nabla^2 E_z^i + k_i^2 E_z^i = 0, \tag{22a}$$

$$\frac{\partial^2 H_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial H_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z^i}{\partial\theta^2} + k_i^2 H_z^i = \nabla^2 H_z^i + k_i^2 H_z^i = 0, \tag{22b}$$

where $k_i = \omega^2\epsilon_i\mu_i$.

In close analogy with the antiplane problem in linear elasticity, the following separable form for E_z^i and H_z^i can be postulated ($\forall\ (r,\theta)\in\Omega_i$) [12]:

$$E_z^i(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}\ (\theta,\lambda_j), \quad H_z^i(r,\theta) = \sum_j r^{\lambda_j} F_{i,j}\ (\theta,\lambda_j) \tag{23}$$

where λ_j are the eigenvalues, and $f_{i,j}$, and $F_{i,j}$ are the eigenfunctions.

We can introduce Eq. (23) into Eq. (22), obtaining the following equalities:

$$r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2}+\lambda_j^2 f_{i,j}\right)=0, \tag{24a}$$

$$r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 F_{i,j}}{\mathrm{d}\theta^2}+\lambda_j^2 F_{i,j}\right)=0, \tag{24b}$$

Hence, we find that the eigenfunctions $f_{i,j}$ and $F_{i,j}$ are linear combinations of trigonometric functions, in perfect analogy with the eigenfunction $f_{i,j}$ in antiplane elasticity (see Eq. (8)):

$$f_{i,j}(\theta,\lambda_j)=A_i\sin(\lambda_j\theta)+B_i\cos(\lambda_j\theta), \tag{25a}$$

$$F_{i,j}(\theta,\lambda_j)=C_i\sin(\lambda_j\theta)+D_i\cos(\lambda_j\theta). \tag{25b}$$

These eigenfunctions are responsible for the singular behaviour of the components E_r^i, E_θ^i, H_r^i and H_θ^i of the electric and magnetic fields near the wedge apex. In particular, from Eq. (21), we observe that:

$$E_r^i=\frac{1}{r\mathrm{j}\omega\epsilon_i}\frac{\partial H_z^i}{\partial\theta}=\frac{1}{\mathrm{j}\omega\epsilon_i}\sum_j\lambda_j r^{\lambda_j-1}F'_{i,j}\sim O(r^{\lambda_j-1}), \tag{26a}$$

$$E_\theta^i=-\frac{1}{\mathrm{j}\omega\epsilon_i}\frac{\partial H_z^i}{\partial r}=-\frac{1}{\mathrm{j}\omega\epsilon_i}\sum_j\lambda_j r^{\lambda_j-1}F_{i,j}\sim O(r^{\lambda_j-1}), \tag{26b}$$

$$H_r^i=-\frac{1}{r\mathrm{j}\omega\mu_i}\frac{\partial E_z^i}{\partial\theta}=-\frac{1}{\mathrm{j}\omega\mu_i}\sum_j\lambda_j r^{\lambda_j-1}f'_{i,j}\sim O(r^{\lambda_j-1}), \tag{26c}$$

$$H_\theta^i=\frac{1}{\mathrm{j}\omega\mu_i}\frac{\partial E_z^i}{\partial r}=\frac{1}{\mathrm{j}\omega\mu_i}\sum_j\lambda_j r^{\lambda_j-1}f'_{i,j}\sim O(r^{\lambda_j-1}). \tag{26d}$$

Hence, $E_z^i\sim O(r^{\lambda_j})$ and $H_z^i\sim O(r^{\lambda_j})$ are the analogous counterparts of u_z^i and remain finite for $r\to 0$. Moreover, the radial components of the electric and magnetic fields, E_r^i and H_r^i, are analogous to $\tau_{\theta z}^i$ and the circumferential components, E_θ^i and H_θ^i, are analogous to τ_{rz}^i. More specifically, we have $E_r^i=\tau_{\theta z}^i/(\mathrm{j}\omega\epsilon_i G_i)$, $H_r^i=-\tau_{\theta z}^i/(\mathrm{j}\omega\mu_i G_i)$, $E_\theta^i=-\tau_{rz}^i/(\mathrm{j}\omega\epsilon_i G_i)$ and $H_\theta^i=\tau_{rz}^i/(\mathrm{j}\omega\mu_i G_i)$. All of these components diverge when $r\to 0$ with a power-law singularity of order $-1<(\lambda_j-1)<0$.
Regarding the BCs, the tangential components of the electric field vanish along the edges Γ_1 and Γ_n of the PEC:

$$E_z^1(r,\gamma_1)=0,\quad E_r^1(r,\gamma_1)=0, \tag{27a}$$

$$E_z^{n-1}(r,\gamma_n)=0,\quad E_r^{n-1}(r,\gamma_n)=0, \tag{27b}$$

On the PEC surface also $H_\theta=0$, but this condition needs not be enforced, since it is a consequence of the previous ones. Along each bi-material interface ($i=1,\ldots,n-2$), the tangential components of the electric and magnetic fields are continuous, i.e.

$$E_z^i(r,\gamma_{i+1})=E_z^{i+1}(r,\gamma_{i+1}),\quad E_r^i(r,\gamma_{i+1})=E_r^{i+1}(r,\gamma_{i+1}), \tag{28a}$$

$$H_z^i(r,\gamma_{i+1})=H_z^{i+1}(r,\gamma_{i+1}),\quad H_r^i(r,\gamma_{i+1})=H_r^{i+1}(r,\gamma_{i+1}). \tag{28b}$$

Using Eqs. (26), the BCs (27) become:

$$E_z^1(r,\gamma_1)=0, \tag{29a}$$

$$E_z^{n-1}(r,\gamma_n)=0, \tag{29b}$$

$$\frac{\partial H_z^1}{\partial\theta}(r,\gamma_1)=0, \tag{29c}$$

$$\frac{\partial H_z^{n-1}}{\partial\theta}(r,\gamma_n)=0, \tag{29d}$$

whereas those defined by Eq. (28) become $(i = 1, \ldots, n-2)$:

$$E_z^i(r, \gamma_{i+1}) = E_z^{i+1}(r, \gamma_{i+1}), \tag{30a}$$

$$\frac{1}{\epsilon_i}\frac{\partial H_z^i}{\partial \theta}(r, \gamma_{i+1}) = \frac{1}{\epsilon_{i+1}}\frac{\partial H_z^{i+1}}{\partial \theta}(r, \gamma_{i+1}) \tag{30b}$$

$$H_z^i(r, \gamma_{i+1}) = H_z^{i+1}(r, \gamma_{i+1}), \tag{30c}$$

$$\frac{1}{\mu_i}\frac{\partial E_z^i}{\partial \theta}(r, \gamma_{i+1}) = \frac{1}{\mu_{i+1}}\frac{\partial E_z^{i+1}}{\partial \theta}(r, \gamma_{i+1}). \tag{30d}$$

It is interesting to note that Eqs. (22), (29) and (30) can be separated into two independent sets of equations, one involving only H_z and another involving only E_z. Hence, the electromagnetic field for this problem can be decomposed into two distinct independently evolving fields, the so-called Transverse Electric (TE) and Transverse Magnetic (TM) fields, respectively. In particular, the TE (resp. TM) field has vanishing electric (resp. magnetic) but nonzero magnetic (resp. electric) field parallel to the cylinder axis z.

Considering the series expansion for E_z and H_z, along with the expressions for the eigenfunctions $f_{i,j}$ and $F_{i,j}$, the boundary value problem consists of two sets of $2n-2$ equations in $2n-1$ unknowns, one for E_z and another for H_z. The former equation set (TM case) involves the coefficients $A_{i,j}$, $B_{i,j}$ and λ_j and can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{v} = \mathbf{0}, \tag{31}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue and $\mathbf{v}$ represents the vector which collects the unknowns $A_{i,j}$ and $B_{i,j}$. The coefficient matrix in Eq. (31) has exactly the same structure as that for the elasticity problem in Eq. (13), provided that we consider $\mathbf{N}_\theta^i = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}$ and we set $G_i = 1/\mu_i$.

The latter equation set (TE case) involves the coefficients $C_{i,j}$, $D_{i,j}$ and λ_j and can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{w} = \mathbf{0}, \tag{32}$$

where $\mathbf{\Lambda}$ is the coefficient matrix which depends on the eigenvalue and $\mathbf{w}$ represents the vector which collects the unknowns $C_{i,j}$ and $D_{i,j}$. Again, the coefficient matrix in Eq. (32) has exactly the same structure as that for the elasticity problem in Eq. (13), provided that we consider $\mathbf{N}_\theta^i = \{\cos(\lambda_j\theta), -\sin(\lambda_j\theta)\}$ and we set $G_i = 1/\epsilon_i$.

For the existence of nontrivial solutions, the determinants of the coefficient matrices must vanish, yielding two eigenequations that, for given values of ϵ_i and μ_i, determine the eigenvalues λ_j^{TE} and λ_j^{TM}. Hence, this proves that the analysis of the singularities of the electro-magnetic field is mathematically analogous to that for the elastic field due to antiplane loading.

CONCLUSIONS

In the present paper, we have compared and unified the mathematical formulations for the asymptotic characterization of the singular fields at multi-material wedges in antiplane elasticity, diffusion problems and electromagnetims. The asymptotic analysis of the stress-singularities at the vertex of multi-material wedges and junctions in antiplane elasticity is perfectly analogous to the corresponding diffusion problem. The temperature field is analogous to the out-of-plane displacement field and the heat fluxes are analogous to the tangential stresses. On the other hand, the analogy with electromagnetism is more complex. In particular, when an isotropic multi-material wedge with PEC boundaries is considered, we have shown that two independent problems can be defined, one for TE fields, associated to an eigenequation for H_z, and one for TM fields, associated to an eigenequation for E_z. The eigenequation for E_z corresponds exactly to that obtained for the same geometrical configuration in antiplane elasticity by setting

$G_i = 1/\mu_i$ and replacing the PEC region with an infinitely stiff material leading to clamped edge BCs along Γ_1 and Γ_n. Similarly, the other eigenequation for H_z can be obtained in antiplane elasticity for the same geometrical configuration by setting $G_i = 1/\epsilon_i$ and replacing the PEC region with an infinitely soft material leading to stress-free BCs along Γ_1 and Γ_n.

REFERENCES

[1] Sinclair, G.B.: Stress singularities in classical elasticity-I: removal, interpretation, and analysis, Appl. Mech. Rev. (2004) Vol. 57, pp. 251–297.

[2] Sinclair, G.B.: Stress singularities in classical elasticity-II: asymptotic identification, Appl. Mech. Rev. (2004) Vol. 57, 385–439.

[3] Paggi, M.; Carpinteri, A.: On the stress singularities at multimaterial interfaces and related analogies with fluid dynamics and diffusion, Appl. Mech. Rev. (2008) Vol. 61, pp. 1–22.

[4] Rao, A.K.: Stress concentrations and singularities at interface corners, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM) (1971) Vol. 51, pp. 395–406.

[5] Fenner, D.N.: Stress singularities in composite materials with an arbitrarily oriented crack meeting an interface, Int. J. Fract. (1976) Vol. 12, pp. 705–712.

[6] Williams, M.L.: Stress singularities resulting from various boundary conditions in angular corners of plates in extension, J. Appl. Mech. (1952) Vol. 74, pp. 526–528.

[7] Ma, C.C.; Hour, B.L.: Antiplane problems in composite materials with an inclined crack terminating at a bi-material interface, Int. J. Sol. Struct. (1990) Vol. 26, pp. 1387–1400.

[8] Pageau, S.S.; Joseph, P.F.; Biggers Jr., S.B.: Finite element evaluation of free-edge singular stress fields in anisotropic materials, Int. J. Num. Meth. Engng. (1995) Vol. 38, pp. 2225–2239.

[9] Sinclair, G.B.: On the singular eigenfunctions for plane harmonic problems in composite regions, J. Appl. Mech. (1980) Vol. 47, pp. 87–92.

[10] Paggi, M.; Carpinteri, A.; Orta, R.: A mathematical analogy and a unified asymptotic formulation for singular elastic and electromagnetic fields at multimaterial wedges, Journal of Elasticity, in press. doi:10.1007/s10659-009-9236-y

[11] Bouwkamp, C.: A note on singularities occurring at sharp edges in electromagnetic diffraction theory, Physica (1946) Vol. 12, p. 467.

[12] Meixner, J.: The behavior of electromagnetic fields at edges, IEEE Trans. Antennas Propag. (1972) AP20, pp. 442–446.

[13] Carpinteri, A.; Paggi, M.: Asymptotic analysis in Linear Elasticity: From the pioneering studies by Wieghardt and Irwin until today, Engng. Fract. Mech. (2009) Vol. 76, pp. 1771–1784. Invited paper presented at the Karl Wieghardt & George R. Irwin Centenary Conference on Structural Integrity in the Service of Public Safety, Vienna, Austria, 2007.

[14] Dean, W.R.; Montagnon, P.E.: On the steady motion of viscous liquid in a corner, Proc. Cambridge Philos. Soc. (1948) Vol. 45, pp. 389-395.

[15] van Bladel., J.: Singular Electromagnetic Fields and Sources, Clarendon Press, Oxford, UK (1991).

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