FRACTURE MECHANICS FOR MEMBRANES

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Abstract

During fracture of membranes loading often produces buckles above and below the crack surface. This changes the stress state surrounding the crack-tip and stresses in the neighbourhood of the crack-tip posses a weaker singularity than $r^{-1/2}$. As a result, fracture occurs when the crack-tip stress distribution is different as compared with that when buckling is artificially prevented. Therefore the conditions for transfer of lab results to real structures are changed. The weaker singularity is here utilised to formulate an adopted fracture mechanical theory. An approximate application is made based on an assumption that the buckled area of the paper is incapable of carrying load. This region is approximated with the region that is under compressive load at plane stress conditions. The result is compared with experiments performed on paper. The importance of the linear extent of the process region has on the energy available for fracture is discussed.

Introduction

Fracture mechanical theories usually assume that the fractured specimen remain in its original plane. Fracture in thin sheets often occurs in the presence of displacements perpendicular to the original plane of the sheet either due to motion of the entire sheet, which results in a tearing fracture mode or due to local buckling of the sheet, usually around the crack surfaces. In this paper a fracture mechanical theory that recognises buckling is proposed.

The motivation for the interest fracture of thin sheets is its frequent appearance in pipes, balloons or structures with skin, like aircrafts, boats and cars. For example, a crack running along a pressurized pipe will be subject to tensile stress ahead of the crack and either compressive loads above and below the crack surfaces or undulating small bending stresses accompanied by buckling. This is also the case when the hull of a ship is dented and torn at grounding. In nuclear pressure vessels large plate structures are arranged to control or to improve the water flow in the reactor. Small fatigue cracks in these lead to costly and possibly unnecessary repairing often because the risk of failure cannot be predicted due to gross buckling and tearing.

Buckling in connection with fracture of thin sheets/members has received some attention [1-3]. However, in view of the practical importance far too little is understood of the effect of buckling on the fracture toughness. Several reasons are obvious. First the mechanical state is difficult to compute. A three-dimensional analysis for large displacements is generally needed. For thin sheets, buckling loads are much smaller than the loads at fracture and higher loads usually lead to multiple buckling. This may cause numerical difficulties. Second the state of the fracture process region may be affected by the out of plane shearing accompanying the buckling.

Our aim is to increase the fundamental understanding of fracture of membranes. Thus, an attempt to create a connection between testing of paper and prediction of failure loads a linear fracture mechanical theory is developed based on a perception of buckling as a weakening of the sheet. Here an asymptotic field surrounding a crack-tip is computed presuming that the load carrying capacity of the buckling parts of the sheet is insignificant. The buckling parts are considered to be stress free, which leads a modification of the boundary value problem posed by traction free crack surfaces. The theory is examined in view of experiments on paper.

Theory

At small scale yielding the stress and strain field surrounding crack-tips is known. The field surrounds the crack-tip both in a test specimen and in an engineering structure. This is the basis for testing and prediction of failure loads in real cases. The only variable of the local load is the strength of the field. At characteristic events, such as crack growth initiation, fatigue at a specified rate or whatever could be thought of the strength of the crack-tip field is the same in the test specimen as in any other structure.

The stress field in the body is observed in two magnification levels. An outer field gives the stresses in the body and an inner field gives the near crack-tip stresses. An expansion of the outer field has a leading term that dominates as the distance to the crack tip decrease, whilst the plastic zone is regarded to be small. On the other side, with the focus on an inner field on the scale of the extent of the non-linear region surrounding the crack-tip, the stress field is affected by deviation from linear elasticity caused by non-linear material behaviour during plastic deformation and the material degradation preceding fracture. Outside the non-linear region stresses can be expanded in a series with the same leading term as mentioned before for the outer field, but now this term dominates as the distance become large, *i.e.*, large in relation to the linear extent of the non-linear region. If the non-linear region is small enough there is an annular region where the stresses are uniquely given by the strength of a single term stress field defined by its amplitude. The inner region or the crack-tip processes is totally cut out from any influence of remote load apart from this amplitude.

To advance this, a thin sheet of finite size containing a crack is considered. The sheet is subjected to tensile stress normal to the plane of the crack as shown in Fig. 1. A Cartesian coordinate system is placed in the sheet with the origin in the centre of the sheet and a polar coordinate system, r and θ, is attached to the crack tip at $x = a$ and $y = 0$, defined with $\theta = 0$ in the direction of the positive x-axis. The stress field in the vicinity of the crack tip may then be written as follows:

$$\sigma_{ij} = k\,\sigma_o \left(\frac{r\,\sigma_o^2}{K_{\mathrm{I}c}^2} \right)^{-s} f_{ij}(\theta) + \mathrm{O}(r/a)^0 + \mathrm{O}(r_p/r)\,, \quad \text{as } r/a \to 0 \text{ and } r_p/r \to 0, \qquad (1)$$

where r_p is the linear extent of the non-linear region at the crack-tip. The non-dimensional coefficient k is the amplitude of the stress field. The angular functions f_{ij} are known. The distance from the crack-tip is given in relation to two material parameters, the fracture toughness, K_{Ic}, and a characteristic stress, σ_o, *e.g.*, the yield stress.

Should the specimen not buckle the expression become reduced to the stresses of linear elastic fracture mechanics by putting $k = K_{\mathrm{I}}/(\sqrt{2\pi K_{\mathrm{I}c}})$, observing that in this case $s = 1/2$. The result for linear fracture mechanics is based on a calculation of a boundary value problem with traction free crack surfaces of the leading term.

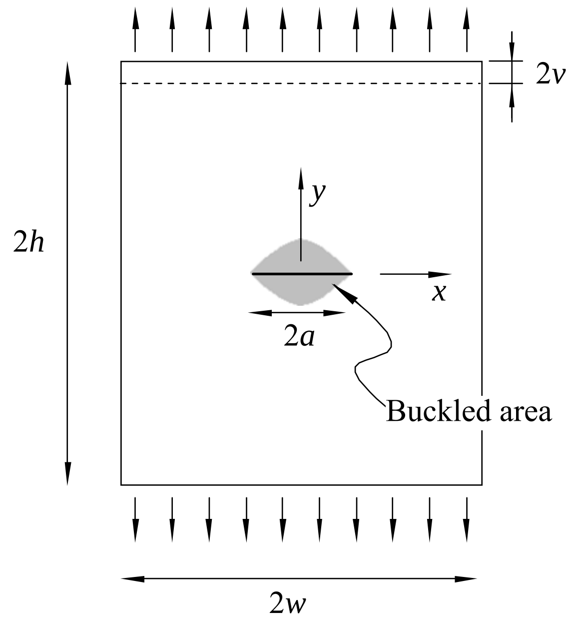

FIGURE 1. The sheet with central crack and the expected buckled region.

Calculation of a crack that remain in plane show that stresses are compressive above and below the crack, close to the crack surface. These stresses are relaxed if the sheet buckles. In this preliminary analysis we assume that the region with compressive stresses, when plane stress conditions are assumed, coincide with the buckled region for a real case. Further the analysis assumes that the buckled region around the crack has negligible load carrying capability. Therefore in the model developed here the stresses in the sheet are regarded to be as if the material of buckled region around the crack were removed leaving an opening larger than the crack.

This affects the exponent s in (1). The shape of the buckled area near the crack-tip provides an angle, θ_o, which is used to calculate s (see Fig. 2). The boundary conditions in the vicinity of the crack tip are the traction free boundaries of the notch that remains after removal of the material in the compressed and supposedly buckled region. These boundaries are found at $\theta = \pm\theta_o$.

Letting stresses be represented by the expressions (cf. *e.g.* Broberg [4])

$$\sigma_{rr} = \frac{\partial^2 \Phi}{r\partial r} + \frac{\partial^2 \Phi}{r^2 \partial\theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} \quad \text{and} \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \Phi}{r\partial\theta}\right) \quad , \tag{2}$$

where the Airy stress function, Φ, satisfies the biharmonic equation

$$\Delta\Delta\Phi = 0 \quad . \tag{3}$$

By assuming an Airy stress function of the following form

$$\Phi = r^{2-s}\left\{A\cos\left[(s-2)\theta\right] + B\cos(s\theta)\right\} \quad , \tag{4}$$

the boundary conditions

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad , \tag{5}$$

for traction free notch edges give the following equation

$$(s-1)(s-2)\{s\sin s\theta_o \cos(s-2)\theta_o - (s-2)\sin(s-2)\theta_o \cos s\theta_o\} = 0 \quad . \tag{6}$$

This equation generates all admissible exponents *s*. For the cases investigated only one real root was found in the interval $0 < s < 1$.

The buckled region is computed using finite element method ABAQUS [5]. The angle θ_o is determined by observation of the border across which the pressure switches sign. This is shown schematically in Fig. 2.

Since the crack-tip singularity, *s*, depend on the extent of buckling and is usually different from 1/2 there will not be any stress intensity factor defined in a conventional way (cf. Williams [6]). Here the stress intensity factor K_{I} is defined as

$$K_{\mathrm{I}} = \sigma_\infty \sqrt{\pi a}\; f\left(\frac{a}{w}, \frac{h}{w}\right) \quad , \tag{7}$$

where σ_∞ is the uniaxial stress at infinity, and *a*, *h* and *w* are length parameters according to Fig. 1. The stress σ_∞ is calculated from prescribed displacement, *v*, as $\sigma_\infty = (v/h)\,E$. The function *f* is used to compensate for finite specimen dimensions (cf. Isida [7]). Fracture toughness $K_{\mathrm{I}c}$ is defined as the stress intensity at onset of crack growth.

The relation

$$\sigma_{ij} = \sigma_o (r/r_p)^{-s} f_{ij} \quad , \tag{8}$$

give the stress in the annular ring dominated by the single term given by the expression (1). Here expressed as a function of parameters specified for the non-linear near tip region. The relation

$$\sigma_{ij} = \sigma_\infty (r/a)^{-s} f_{ij} \quad , \tag{9}$$

Give the same stresses as in (8) expressed using remote stress field and specimen geometry. Expected autonomy of the non-linear region at the crack-tip implies that this region is identical and independent of crack length at, *e.g.*, onset of crack growth. Therefore

$$\sigma_\infty^{(a_o)} (r/a_o)^{-s} = \sigma_o (r/r_p)^{-s} = \sigma_\infty^{(a)} (r/a)^{-s} \quad , \tag{10}$$

The meaning of this become clear if the critical stresses (at onset of crack growth) $\sigma_\infty^{(a_o)}$ for half crack length a_o and $\sigma_\infty^{(a)}$ at half crack length *a* are inserted into (7) to compute fracture toughnesses, $K_{\mathrm{I}c} = K_{\mathrm{I}}$. The result can be written

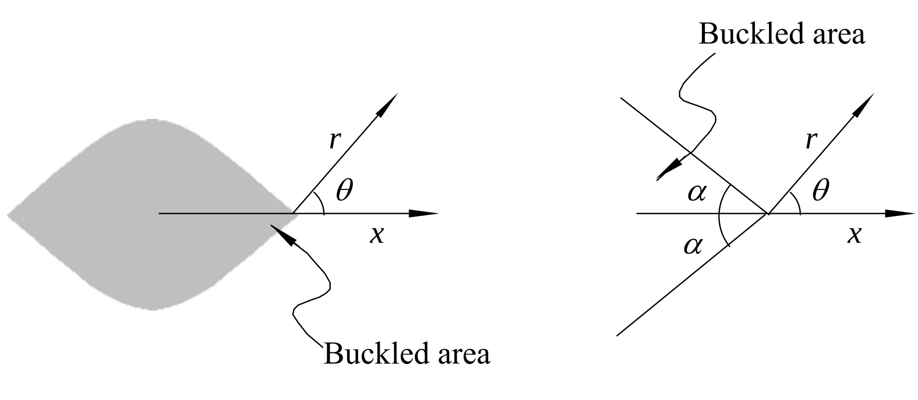

FIGURE 2. Enlarged view of the crack-tip.

$$K_{\mathrm{I}c}^{(a)} = \sigma_\infty^{(a)}\sqrt{\pi a}\ f(\frac{a}{w},\frac{h}{w}) = \left(\frac{a}{a_o}\right)^{1/2-s} \frac{f(a/w, h/w)}{f(a_o/w, h/w)} K_{\mathrm{I}c}^{(a_o)} \quad , \qquad (11)$$

where $K_{\mathrm{I}c}^{(a)}$ is the fracture toughness for a half crack length a and $K_{\mathrm{I}c}^{(a_o)}$ dito for a half crack length a_o. For a large specimen for which $a << w$ and $h << w$ this becomes

$$K_{\mathrm{I}c}^{(a)} = (a/a_o)^{1/2-s} K_{\mathrm{I}c}^{(a_o)} \quad . \qquad (12)$$

Assume that material testing is performed for the crack length a_o. Since buckling leads to a weaker stress singularity, i.e., $s < 1/2$ (12) shows that the expected toughness according to (7) increases with increasing crack length. Not considering buckling would then underestimate the strength of a membrane if testing is performed on specimens with small cracks whereas the opposite holds for predictions for specimens smaller than the tested samples.

Finite element simulations and results

Simulations are performed for two crack types, edge cracks and central cracks. The material is regarded as linear elastic described by the modulus of elasticity and Poisson's ratio. Plane stress is assumed. The sheet is subjected to displacement load as shown in Fig. 1. Along the edges where load is applied the sheet is restrained from movement in the x direction. The two edges perpendicular to the crack plane are free along its entire length. For edge cracks one half of the sheet is simulated and for central cracks one quarter with proper boundary conditions along symmetry lines.

Buckling in the vicinity of a crack-tip is confined to a region $\pi \leq \theta \leq \theta_o$. The angle θ_o is obtained virtually by observing the zero pressure contour of the finite element result. The results are given as

$$\alpha = \pi - \theta_o \quad . \qquad (13)$$

The resulting angles α are given in Table 1 for different geometries. For a central crack, α decreases with increasing crack length, a. The limit for small cracks is of the order of 40°. For central cracks larger than say 0.2 of the specimen width, α seem to be 10° - 12°. For edge cracks α is of the order of 16° in the limit for small cracks. For larger cracks when a approach w the buckling region is confined to α around 10°.

The exponents s corresponding to these angles α are calculated using (6) and (13). Only roots in the interval $0 < s < 1$ are considered. Table 1 show the results. The variation is observed to be quite large whereas s may be as small as 0.15, *e.g.*, for a small central crack. As expected all geometries give a weaker singularity than the one for in-plane linear fracture mechanics, *i.e.*, $s < 0.5$, except for when no buckling occurred in which case s is 0.5.

Using the calculated singularity it is possible to establish a relation between toughnesses defined by the stress intensity factor of (7). The relation (12) is used to compute $K_{\mathrm{I}c}^{(a_o)}$, $K_{\mathrm{I}c}^{(2a_o)}$ and $K_{\mathrm{I}c}^{(10a_o)}$ for crack lengths $a = a_o$, $2a_o$ and $10a_o$ respectively. The ratios a/w and h/w are kept constant. The results displayed in Table 2 show the ratios $K_{\mathrm{I}c}^{(a_o)}/K_{\mathrm{I}c}^{(2a_o)}$ and $K_{\mathrm{I}c}^{(a_o)}/K_{\mathrm{I}c}^{(10a_o)}$. As observed differences may be very large. However if very small central cracks are excluded the relative differences in $K_{\mathrm{I}c}$ are below 10% when a double size specimen is tested. If specimen size is increased 10 times (apart from thickness) the prediction is that the increase of $K_{\mathrm{I}c}$ may be up to two times larger for a small centred crack and 1.35 times larger for an edge crack.

TABLE 1. Notch opening angles and stress singularities. For all geometries *h*/*w* = 1.

	Central crack		Edge crack	
a/w	α	*s*	α	*s*
0.010	40°	0.157	16°	0.370
0.050	29.1°	0.278	11.9°	0.424
0.100	16.5°	0.390	5.9°	0.465
0.200	12.1°	0.423	4.5°	0.474
0.300	11.8°	0.425	5.7°	0.466
0.400	12.2°	0.422	5.4°	0.468
0.500	9.3°	0.443	5.8°	0.465
0.667	10°	0.438	6.8°	0.459
0.833	10.7°	0.433	9.4°	0.442

TABLE 2. Theoretical fracture toughness as function of crack length. For all geometries *h*/*w* = 1.

	Central crack			Edge crack		
a/w	*s*	$K_{\mathrm{I}c}^{(a_o)} / K_{\mathrm{I}c}^{(2a_o)}$	$K_{\mathrm{I}c}^{(a_o)} / K_{\mathrm{I}c}^{(10a_o)}$	*s*	$K_{\mathrm{I}c}^{(a_o)} / K_{\mathrm{I}c}^{(2a_o)}$	$K_{\mathrm{I}c}^{(a_o)} / K_{\mathrm{I}c}^{(10a_o)}$
0.010	0.157	1.268	2.203	0.370	1.094	1.350
0.050	0.278	1.166	1.667	0.424	1.054	1.191
0.500	0.443	1.040	1.140	0.465	1.025	1.084
0.667	0.438	1.044	1.153	0.459	1.029	1.100
0.833	0.433	1.048	1.167	0.442	1.041	1.143

Experiments

A series of experiments of considerably large specimens were performed to find support for the proposed theory. Specimens are chosen fairly large to avoid the complications because of the finite dimensions of the non-linear region surrounding the crack tip. By choosing the specimens large it is believed the effects of non-linear material behaviour can be avoided.

An in house designed testing machine was used. A complete description of the experimental set up and the mechanical conditions applied in the experiment is given in Espinosa [8]. Paper samples 2000 mm long and 1500 mm wide were used. The paper was cut from a 50 x 1.5 m^2 brown paper roll, with a paper thickness of 0.1 mm. Buckling with the appearance as in Fig. 3 was observed during loading (the photography shows buckling of tested aluminium foil).

Crack lengths, $2a$, varied from 50 mm to 1400 mm. The results are plotted in Fig. 4. A large scatter is observed. It can be observed how the results follow the same trend. There is a clear increase of K_{Ic} with increasing crack length. A theoretical curve for α between 12° and 15° giving $s = 0.4$ is included in the figure.

Discussion

The buckling gives rise to redistribution of the stresses in the neighbourhood of the crack-tip. The theory based on approximate estimation of the extent of the buckling region around cracks is used. The estimation of the extent of the buckling region is done by observing the compressed region that develops at plane stress. The focus is on the buckling in the vicinity of the crack-tip. Stresses may here be decomposed into a square root singular term, a constant stress, the so called T-stress and other second order stress terms. Because singular stress dominate close to the crack-tip it becomes obvious that the angle θ_o limiting the buckling region has to approach 180° as the crack tip is approached. However this is not visible on a reasonable length scale and there is no reason to go closer to the crack tip than the extent of the non-linear region surrounding the crack-tip. The proposed procedure is motivated for, e.g., paper since material behaviour for this materials is non-linear at least a few mm^2 around the crack-tip. Engineering cracks in paper may range from a few cm to a meter. This gives a ratio of linear extent of non-linear region to crack length of 1/1000 which we believe to cover in the finite element analysis.

There may be doubts as regards the influence of the non-linear region for small cracks and cracks approaching the width of the specimen. These effects were however ruled out when it was observed that the short crack deviation from linear behaviour was not changing when the size of the specimen was changed.

In the present paper the implications of assumed linear fracture mechanics (7) was investigated. However, it is obvious that a modified fracture mechanical theory, when needed, should be based on the amplitude k (see (1)) and its critical value at crack initiation.

Conclusions

An exact theory for fracture of buckling membranes is proposed. Some approximations are made to perform a simplified analysis. The buckling region is never calculated. Instead this region is approximated with the region having compressive load during loading in plane stress. The buckling region does not carry any significant load. Here the region is assumed not to carry any load. Therefore, a singular stress field weaker than the square root singular stress field resides in the crack-tip vicinity. The strength of this field determines the toughness of the membrane as a function of the crack length.

The following conclusions can be drawn

- Buckling affects the fracture toughness

- Experiments give support to the theory

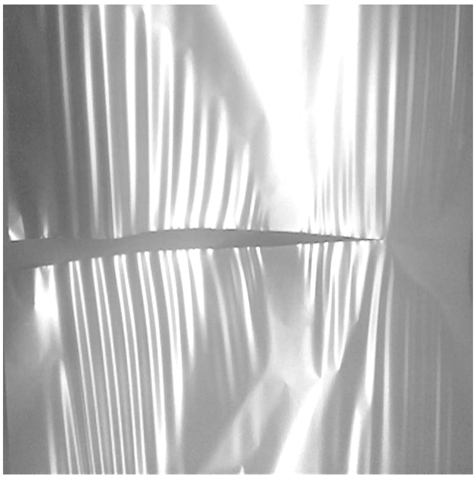

FIGURE 3. Buckling pattern

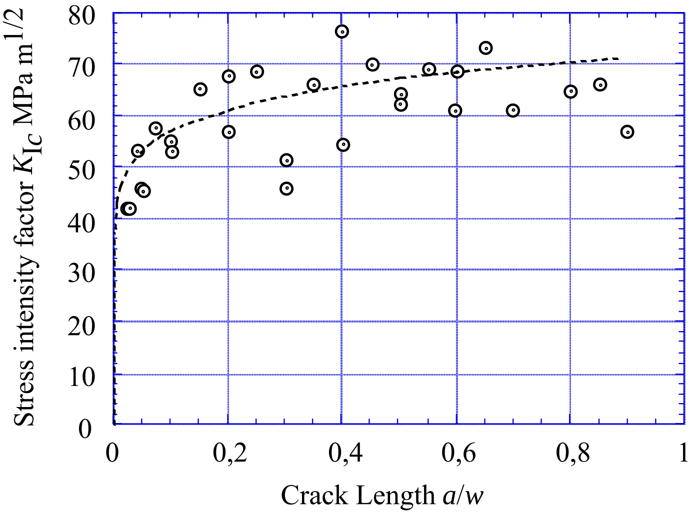

FIGURE 4. K_{Ic} results for the different tests. Dashed curve shows the theoretical result for $s = 0.4$.

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