Effect of Stress Singularities on Scaling of Quasibrittle Fracture

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Abstract Modern engineering structures are often made of quasibrittle materials, which are brittle and heterogeneous. Typical examples include concrete, fiber composites, woven composites, tough ceramics, and nano-composites. The salient feature of quasibrittle structures is that the size of the fracture process zone is not negligible compared to the structure size, which leads to an intricate size effect on the structural strength. The current understanding of scaling of quasibrittle fracture is limited to structures with either strong stress singularities or zero stress singularities. Nevertheless, many engineering structures are designed to have complex geometries, which could cause weak stress singularities. This paper investigates the effect of stress singularities on the scaling of quasibrittle fracture both analytically and numerically. The theoretical analysis is derived from a generalized weakest link model where the energetic scaling of quasibrittle fracture is incorporated into the classical finite weakest link model. The proposed model yields a general scaling equation, which captures the transition from the energetic scaling to statistical scaling as the stress singularity gets weaker. The proposed analytical model is then verified by a numerical study on the fracture of concrete beams with a V-notch under three-point bending, where a wide range of notch angles representing different orders of stress singularities is considered.

Keywords Size Effect, Deterministic Analysis, Finite Weakest Link Model, Quasibrittle Materials.

1. Introduction

Many large-scale engineering structures, such as bridges, dams, aircraft and ships, are usually designed by extrapolating the results of small-scale laboratory experiments. In order to correctly perform this design extrapolation, it is of paramount importance to understand the scale effect on the structural strength. This study focuses on structures that are made of brittle heterogeneous (quasibrittle) materials, which include concrete, fiber composites, tough ceramics, rocks, sea ice, etc. For two-dimensional problems, the nominal structural strength is usually defined as $\sigma_N = cP_{\max}/bD$, where $P_{\max}$ = load capacity of structure, D = characteristic structural size to be scaled, b = width of the structure in the transverse direction, and c = constant which could be chosen such that σ_N represents some familiar parameter such as the maximum stress in the structure in the absence of the stress concentration. It has been demonstrated that the nominal strength of quasibrittle structures is subjected to an intricate size effect. The underlying reason is that for quasibrittle structures the size of the material inhomogenieties is not negligible compared to the structure size, which directly leads to a size-dependent failure behavior [1, 2]. So far, two independent mechanisms have been identified to explain the scaling of strength of quasibrittle structures:

1) *Type-1 size effect*: the maximum load of the structure is attained after the stable formation of a large fracture process zone (FPZ) with distributed cracking, which typically occurs in structures with a smooth boundary. The Type-1 size effect for small and medium-size structures is energetic,

which can be derived based on the Taylor expansion of the energy release rate function at zero crack length [1, 2]. The large size asymptote of this type of size effect is governed by the Weibull statistics of material strength. The statistical size effect can be amalgamated with the energetic size effect to form the complete *energetic-statistical* Type-1 size effect [3]. Recent studies also showed that this size effect can be alternatively derived from a finite weakest link model where the structure is statistically represented by a finite chain of representative volume elements (RVEs) and the probability distribution of RVE strength is derived from fracture mechanics of nanocracks propagating by small, activation-energy-controlled, random jumps through a nano-structure [4-7].

2) *Type-2 size effect*: the maximum load of the structure is reached once a single large crack is formed. The Type-2 size effect typically applied to quasibrittle structures containing a large notch or a large stress-free (fatigued) crack formed prior to maximum load. This size effect is purely *energetic* because the fracture must happen at the pre-existing crack tip. The Type-2 size effect can be derived by using the asymptotic approximation of the energy release function for the propagating crack or the J-integral [1,2,8].

It is clear that the Type-1 and Type-2 size effects can be considered as two limiting cases in terms of the order of stress singularity of the structure. The Type-1 size effect law represents the case of zero stress singularity whereas the Type-2 size effect law represents the case of the strongest stress singularity (i.e. "−1/2" stress singularity). Nevertheless, many modern engineering structures are designed with geometric discontinuities, which produce weak stress singularities. There is still a lack of understanding on the transition between these two size effects as a function of the magnitude of the stress singularities. This study aims to formulate a universal size effect equation for quasibrittle structures through both theoretical and numerical investigations on the fracture of structures with a V-notch under mode-I loading.

2. Review of energetic and statistical size effects

Consider a structure of positive geometry containing a V-notch and subjected to mode-I loading (Fig. 1), where the notch angle is denoted by γ. Positive geometry is defined such that the peak load is reached once the fracture process zone (FPZ) is fully developed. Here we further assume that the notch is sufficiently deep, i.e. $\alpha = a/D > 0.1$, where a = notch depth and D = depth of the structure (Fig. 1). In general, the stress concentration at the V-notch tip is governed by two distinct stress singularities, which represent the symmetrical and anti-symmetrical deformation modes [9-11]. For mode-I fracture, only one stress singularity λ prevails, which corresponds to the symmetrical mode. This section briefly reviews the two existing scaling theories, namely energetic and statistical scaling.

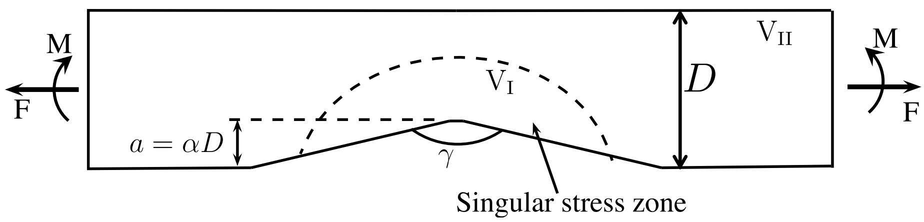

Figure 1. Structure with a V-notch under mode-I facture

2.1. Energetic scaling

Consider the case where the stress singularity is sufficiently strong. It is clear that due to the significant stress concentration the FPZ must form at the notch tip. Therefore, the corresponding scaling mechanism is deterministic. Based on the Williams solution [9], the stress field near the notch tip under mode-I loading can be written as:

$$\sigma_{ij} = Hr^{\lambda} f_{ij}(\theta,\gamma) \tag{1}$$

where r = radial distance from the notch tip, f_{ij} = dimensionless function describing the angular dependence of the stress, and H = stress intensity factor. Based on dimensional analysis, H can be further written as:

$$H = \sigma D^{-\lambda} h(\gamma) \tag{2}$$

where σ = nominal stress $= cP/bD$, P = applied load, and $h(\gamma)$ is the dimensionless stress intensity factor, which is determined by the structural geometry. One commonly used mode-I fracture criterion for structures with a V-notch is that the peak load is attained when the stress at the distance c_f from the notch tip reaches the material tensile strength f_t [12]. Therefore, we obtain the nominal strength of the structure:

$$\sigma_N = f_t \varphi(\gamma)(D/c_f)^{\lambda} \tag{3}$$

where $\varphi(\gamma) = h^{-1}(\gamma) f_{\theta\theta}^{-1}(0,\gamma)$, which can easily calculated by an elastic analysis.

Since Eq. 3 is derived based on the linear elastic fracture mechanics, it represents the large-size asymptote of the energetic size effect law. The small-size asymptote is very easy to construct. For small-size structures, the FPZ occupies the entire notch ligament and consequently the ligament must behave like a crack filled with plastic glue. At this plastic limit, the size effect must vanish. An approximate equation that bridges the small- and large-size asymptotes has been proposed:

$$\sigma_N = \sigma_s \left[1 + (D/D_{0\gamma})^{1/\beta_\gamma}\right]^{\lambda\beta_\gamma} \tag{4}$$

where σ_s = nominal strength at the small-size limit, β_γ = model parameter, and $D_{0\gamma}$ = transitional size $= D_0 \varphi(\gamma)/\varphi(0)$ ($D_0 = D_{0\gamma}$ at $\gamma = 0$). When $\lambda = -1/2$ and $\beta_\gamma = 1$, Eq. 4 converges to the classical Type-2 size effect proposed by Bažant [1, 2], which applies to structures with a large pre-existing crack.

2.2. Statistical size effect

The statistical size effect usually applies to structures without stress singularities, e.g. unnotched beams. A salient feature of the failure of these structures is that the location of damage initiation and localization is uncertain, which is often governed by the randomness of local material strength. Furthermore, these structures reach the peak load once any one of the representative volume elements (RVEs) is damaged and thus the RVE is here defined as the smallest material volume whose failure triggers that failure of the entire structure. The size of RVE l_0 is approximately 2-3 times the size of material inhomogeneities [5]. Statistically speaking, the structure can then be represented by a chain of RVEs. Since the RVE size is about the same as the autocorrelation length of the random material strength field [7], the RVE strength can be treated as an independent random variable and the failure probability of the entire structure can be written as:

$$P_f(\sigma_N) = 1 - \prod_{i=1}^{N} \left[1 - P_1(\sigma_N s(x_i))\right] \tag{5}$$

where P_1 = cumulative distribution function (cdf) of strength of one RVE, N = number of RVEs in the structure, and $s(x_i)$ = dimensionless stress field such that $\sigma_N s(x_i)$ = maximum elastic principal stress at the center of the ith RVE located at x_i. Recent studies have shown that, based on atomistic fracture mechanics and a multiscale transition model for strength statistics, the cdf of RVE strength can be approximated by a Gaussian distribution onto which a Weibull tail is grafted at a probability about 10^{-4} — 10^{-2} [6-7].

The mean strength of the structure can then be calculated as:

$$\bar{\sigma}_N = \int_0^1 \sigma_N \, \mathrm{d}P_f = \int_0^\infty \left[1 - P_f(\sigma_N)\right] \mathrm{d}\sigma_N \tag{6}$$

By considering geometrically similar structures with different sizes, we can obtain the size effect on mean structural strength. Though a closed-form expression seems not possible, an approximate form has been proposed:

$$\bar{\sigma}_N = \left[N_a / D + \left(N_b / D \right)^{rn/m} \right]^{1/r} \tag{7}$$

where n = dimension of scaling and N_a, N_b, and r are constants, which can be determined by the statistical parameters of the cdf of RVE strength. It should be pointed out that Eq. 7 shall not be applied to structures as D approaches 0. A recent study [13] has shown that for small and intermediate-size structures the size effect derived from this finite weakest link model with the use of elastic stresses agrees well with the prediction from the nonlinear deterministic calculation. This is because the mean size effect behavior for small-size and intermediate-size structures is mainly caused by the operative stress redistribution mechanism, which can be well predicted by the nonlinear deterministic calculation. At the same time, this mechanism can also be captured by the finite weakest link model, where the statistical multiscale transition model used for the formulation of the cdf of RVE strength consists of statistical bundles and chains that represent the damage localization and load redistribution mechanisms at different scales (albeit only the elastic stresses are used) [5-7]. For large-size structures, the zone of stress redistribution is negligible compared to structure size and the size effect is mainly caused by randomness of material strength, which cannot be captured by the deterministic calculation. Therefore, the size effect curve for the case of zero stress singularity can be completely explained by the finite weakest link model [4-7].

3. Generalized weakest link model

For structures with a wide V-notch and therefore a weak stress singularity, there is no guarantee that the FPZ would form at the tip of the V-notch. This means that the failure of the structure can be statistically represented by the weakest link model. On the other hand, there exists a singular stress field at the V-notch tip even though the degree of stress concentration is not significant. Furthermore, the fracture of the V-notch itself is associated with an energetic scaling law shown as Eq. 4, which cannot be represented by the existing finite weakest link model. This prompts us to derive a new scaling model by generalizing the classical finite weakest link model to include the energetic scaling of fracture of the V-notch.

In the proposed generalized weakest link model, we isolate the singular stress zone from the remaining part of the structure (Fig. 1), where the singular stress zone can be determined by comparing Eq. 1 to the numerically simulated elastic stress field. Since the singular stress zone is influenced by the presence of the V-notch, whose fracture exhibits an energetic scaling (i.e. Eq. 4), we propose to include this energetic scaling for the calculation of the failure probability of the singular stress zone:

$$P_{f,V_I}(\sigma_N) = 1 - \prod_{i=1}^{N_1}\left\{1 - P_1\left[\mu(D)\sigma_N s(x_i)\right]\right\} \tag{8}$$

where $\mu(D) = \left[1 + (D/D_{0\gamma})^{1/\beta_\gamma}\right]^{-\lambda\beta_\gamma}$, and N_I = number of RVEs in the singular stress zone. The failure probability of the remaining part of the structure can be calculated by the usual weakest link model with the elastic principal stresses:

$$P_{f,V_{II}}(\sigma_N) = 1 - \prod_{i=1}^{N_2}\left\{1 - P_1\left[\sigma_N s(x_i)\right]\right\} \tag{9}$$

where N_2 = number of RVEs in the region outside the singular stress zone. The failure probability of the entire structure can then be written as:

$$P_f(\sigma_N) = 1 - \left[1 - P_{f,V_I}(\sigma_N)\right]\left[1 - P_{f,V_{II}}(\sigma_N)\right] \tag{10}$$

from which we can calculate the mean size effect. Similar to the case of statistical scaling, a closed form expression is impossible. Since the entire framework still relies on the finite weakest link model, we can use the same asymptotic matching technique as that for the statistical scaling to obtain an approximate scaling law:

$$\bar{\sigma}_N = \sigma_0\left\{C_1\left[\mu^m(D)\psi_1 + \psi_2\right]^{-r/m}\left(\frac{D+l_s}{l_0}\right)^{-2/m}\exp[-(\lambda/\lambda_1)^2] + \frac{\mu^{-r}(D)D_b}{\exp[-(\lambda/\lambda_2)^2]D + l_p}\right\}^{1/r} \tag{11}$$

where σ_0 = reference stress, C_1, r, λ_1, λ_2, l_s, l_p, D_b = constants, m = Weibull modulus, $\psi_1 = \int_{V_I}\left\langle s(x)\right\rangle^m \mathrm{d}V(x)$, and $\psi_2 = \int_{V_{II}}\left\langle s(x)\right\rangle^m \mathrm{d}V(x)$. Note that here we introduced l_s and l_p to regularize the functional behavior as D approaches 0. Furthermore, it is easy to show that the large- and small-size asymptotes of the mean strength requires:

$$\sigma_0 C_1^{1/r} = s_0^{1/r}\Gamma(1+1/m) \tag{12}$$

$$\sigma_s = \sigma_0\left\{D_b/l_p + C_1(\psi_1+\psi_2)^{1/m}(l_s/l_0)^{-2/m}\exp[-(\lambda/\lambda_1)^2]\right\}^{1/r} \tag{13}$$

where s_0 = Weibull scaling parameter. The small-size strength limit σ_s can usually be obtained by simple plastic analysis by treating the ligament as a crack filled by the plastic glue.

It is clear that Eq. 11 converges to Eqs. 4 and 7 in the two limiting cases. For the transition between these two limits, the size effect consists of both energetic and statistical components. At the small-size limit, the size effect is mainly governed by the statistical scaling component since the energetic scaling term predicts a weak size effect. At the large-size limit, the scaling is governed by the Weibull statistics modified by an energetic scaling term, which leads to a compound energetic-Weibull statistical scaling. Though the focus of this study is on mode-I fracture, the present framework has been extended to general mixed-mode fracture, which is applicable to bimaterial structures. When dealing with mixed-mode fracture, the energetic scaling term would generally contain two distinct stress singularities [14].

4. Numerical simulation

To verify the proposed analytical model, we investigate the size effect on strength of concrete beams with a V-notch under three-point bending (Fig. 2). The beam has a 1:6 depth-to-span ratio and notch depth is 20% of the beam depth. In the simulation, we consider different notch angles, i.e. γ = 0°, 90°, 120°, 135°, and 170°. For each notch angle, a series of geometrically similar specimens with a size range 1: 2: 4: 8: 16: 64: 128 (the depths of the smallest and largest beams are 37.5 mm and 4.8 m, respectively) is simulated. Based on the Williams solution, these notch angles correspond to the following orders of mode-I singularity: $\lambda = -0.5,\ -0.455,\ -0.3843,\ -0.3264,\ -0.0916$.

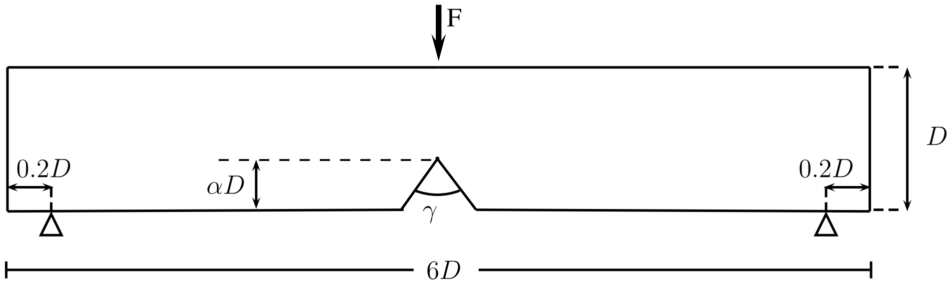

Figure 2. Numerical simulation of three-point bend beam

It is well known that concrete exhibits a complex constitutive behavior. Extensive efforts have been devoted to numerical modeling of fracture of concrete, e.g. [15-18]. Since we are interested in static mode-I fracture, we adopt the default plastic-damage model in ABAQUS because it is sufficient for the purpose of the present study; a detailed description of this constitutive model can be recovered from [19]. The material properties are chosen as follows: Young's modulus E = 30 GPa, Poisson ratio v = 0.2, tensile strength f_t = 3 MPa, compressive strength f_c = 30 MPa, and Mode-I fracture toughness G_f = 100 N/m. Though we specify the compressive strength, the compressive region of the beam is expected to remain elastic. Therefore, the nonlinear part of the compressive behavior is not of particular interest for the present study. All the specimens undergo displacement-controlled loading. In this study, the numerical simulation is performed within a deterministic framework. Previous studies have shown that the deterministic simulation with a strain-softening constitutive model can successfully capture the entire size effect for the case of strong stress singularity and the size effect for the small- and intermediate structure sizes for the case of zero stress singularity [1-2]. Therefore, we expect that for the case of weak stress singularity the deterministic numerical model is sufficient for simulating the size effect for the small-and intermediate structure sizes. For the large-size asymptote, the deterministic simulation cannot yield the statistical scaling components. In this study, we mainly focus on the small- and intermediate size range, which is applicable to most engineering designs. Therefore, only deterministic simulation is necessary. As will be shown later, the influence of the statistical scaling component only prevails in structures of very large size.

For the finite element modeling, the notch tip is considered to have a very small width, i.e. 5 mm, which is a constant for all the geometries and sizes. For the deterministic simulation, the damage occurs near the mid-span of the beam. Therefore, to reduce computational efforts, we model the middle portion of the beam with a refined mesh (i.e. 5 mm) and the damage plasticity model whereas the rest part of the beam is modeled by a coarse mesh with a purely elastic model. For each specimen, the assumed region, where the nonlinear material model is used, is further checked as part of the simulation. As the notch angle increases, this nonlinear region becomes larger. It should be pointed out that the present modeling is not as efficient as the crack band model and the nonlocal model, where larger element sizes can be used. However, the use of the crack band and nonlocal models requires extensive modeling efforts with special cautions such as the choice of crack band width [20] and treatment of the nonlocal weighting function along the structural boundary.

Fig. 3 presents the simulated nominal stress-relative deflection curves for specimens of all sizes and all different notch angles, where the nominal stress is defined as = P/bD and the relative displacement is defined as $\delta = \Delta/D$ (Δ = load-point displacement). It is observed that as the structure size increases the post-peak softening portion of the load-deflection curve becomes steeper, which implies a more brittle failure behavior. It should be noted that for large specimens (i.e. D = 1.2, 2.4, and 4.8m) the post-peak behavior is not captured, which indicates that a snap-back instability may have occurred. The snap-back behavior could be captured by loading the specimens by the crack mouth opening displacement. This is not done because we are interested only in the peak load.

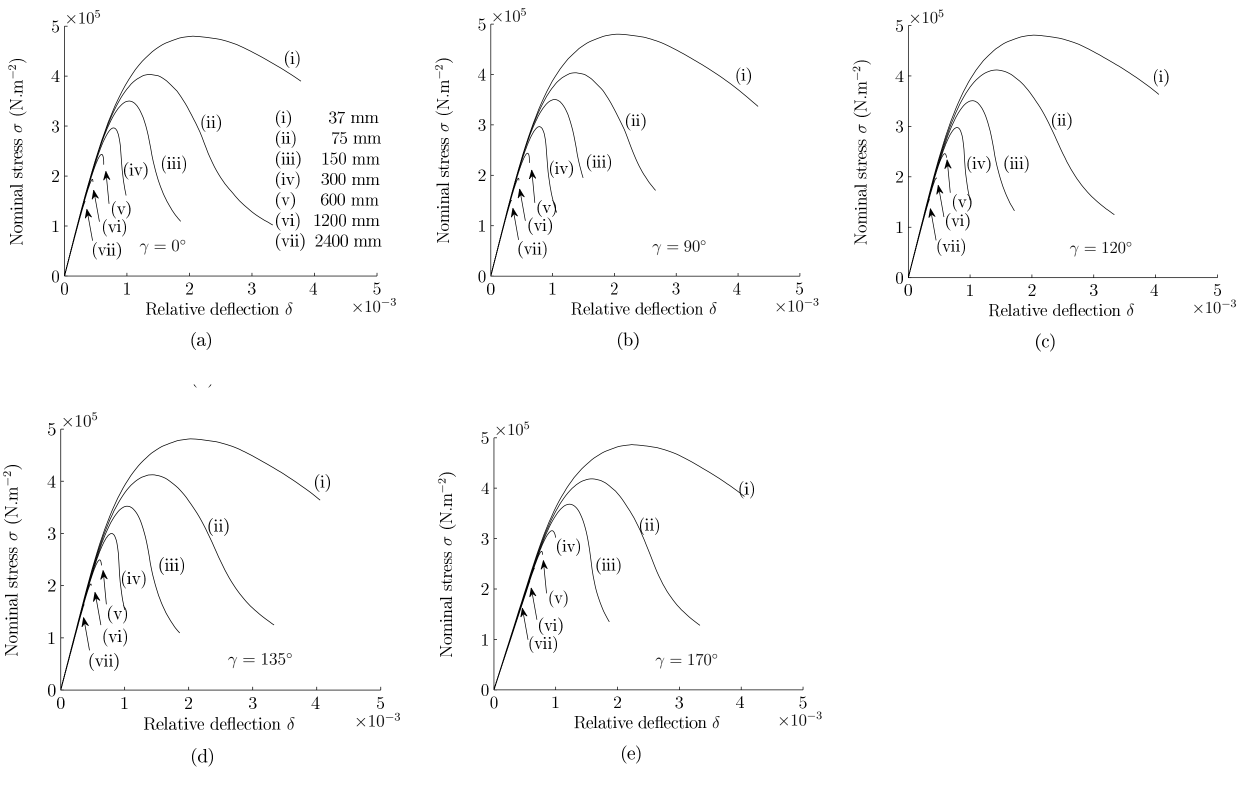

Figure 3. Simulated nominal stress-relative displacement curves

For the 2D specimens, we can simply define the nominal strength of the beam as $\sigma_N = P_{\max} / bD$, where b = 1. Fig. 4 shows the simulated size effects on the nominal strength for different notch angles and the optimum fitting by Eq. 11. As mentioned previously, the small-size strength limit σ_s can easily be calculated by plastic analysis, and the Weibull modulus for concrete is known to be 24. From the fitting, we obtain l_p = 40 mm, l_s = 150 mm, r = 0.88, s_0 = 0.488 MPa, D_0 = 90 mm, λ_1 =

0.301, λ_2 = 0.208, and β_γ =1, 1, 1.5, 2, 1.1 for γ = 0°, 90°, 120°, 135°, and 170°. It is clear from Fig. 4 that the Eq. 11 agrees well with the simulation results. It should be noted that the simulated size effect curve does not match well with Eq. 11 at the large-size limit for beams with a 170° V-notch. This is due to the fact that we used deterministic simulation, which cannot capture the associated large-size asymptote of the classical Weibull scaling relation. Furthermore, it is observed that such a difference occurs for very large beam size (i.e. D > 1.2m), which indicates that deterministic calculation is sufficient for most normal-size concrete beams.

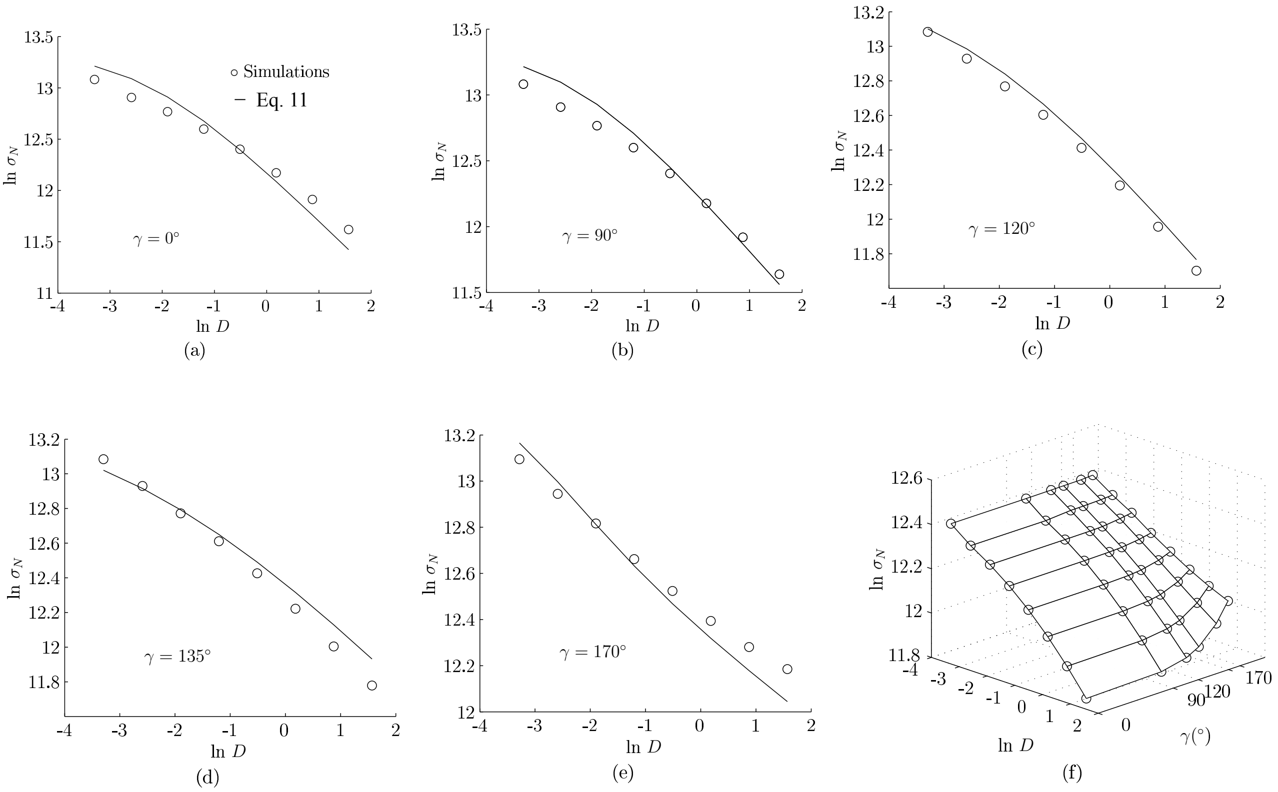

Figure 4. Size effect curves for notched beams: a)-e) Simulated size effect curves fitted by Eq. 11 and f) 3D plot of simulated effects of structural size and notch angle on nominal strength

5. Conclusions

This paper shows that the scaling of strength of quasibrittle structures is strongly dependent on the magnitude of the stress singularities. Such dependence can be derived from a generalized weakest link model, where the classical energetic scaling law is combined with the finite weakest link model. For the case of strong stress singularities, the scaling of fracture is purely energetic, which can be derived from fracture mechanics. For the case of zero stress singularity, the size effect can be explained by random material strength through the finite weakest link model. For the case of weak stress singularities, the scaling is governed by both energetic and statistical mechanisms.

References

[1] Z. P. Bažant, Scaling theory of quasibrittle structural failure. Proc. Nat'l Acad. Sci. USA, 101(37) (2004) 13397–13399.
[2] Z. P. Bažant, Scaling of Structural Strength, 2nd Ed. Elsevier, London, 2005.

[3] Z. P. Bažant, and D., Novák, Energetic-statistical size effect in quasibrittle fracture at crack initiation. ACI Mater. J. 97(3), (2000), 381-392.
[4] Z. P. Bažant, and S.D. Pang, Mechanics-based statistics of failure risk of quasibrittle structures and size effect on safety factors. Proc. Nat'l Acad. Sci. USA, 103(25) (2006) 9434–9439.
[5] Z. P. Bažant, and S.D. Pang, Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. J. Mech. Phys. Solids 55, (2007) 91–134.
[6] Z. P. Bažant, J.-L. Le, and M. Z. Bazant, Scaling of strength and lifetime of quasibrittle structures based on atomistic fracture mechanics. Proc. Nat'l Acad. Sci. USA, 106(28) (2009) 11484–11489.
[7] J.-L. Le, and Z. P. Bažant, Unified Nano-Mechanics Based Probabilistic Theory of Quasibrittle and Brittle Structures: I. Strength, Crack Growth, Lifetime and Scaling. J. Mech. Phys. Solids, 59 (2011), 1291-1321.
[8] Z. P. Bažant, and K. T. Kazemi, Determination of fracture energy, process zone length and brittle number from size effect, with application to rock and concrete. Int. J. Frac. 44, (1990), 111-131.
[9] M. L. Williams Stress singularities resulting from various boundary conditions in angular corners of plates in extension. J. Appl. Mech. 74 (1952), 526-528.
[10] W. C. Carpenter, Mode I and mode II stress intensities for plates with cracks of finite opening. Int. J. Frac., 26, (1984). 201-214.
[11] X. H. Liu, Z Suo, and Q, Ma, Split singularities: Stress field near the edge of silicon die on polymer substrate. Acta Mater., 47, (1999) 67-76.
[12] Z. P. Bažant, and Q. Yu, Size effect on strength of quasibrittle structures with reentrant corners symmetrically loaded in tension. J. Engrg. Mech. 129(11) (2006), 1168-1176
[13] J.-L. Le, J. Elias and Z. P. Bažant, Computation of probability distribution of strength of quasibrittle structures failing at macro-crack initiation. J. Engrg. Mech., ASCE 138(7), (2012) 888–899.
[14] J.-L. Le, General size effect on strength of bi-material quasibrittle structures. Int. J. Frac. 172 (2011), 151-160.
[15] Z. P. Bažant, F. C. Caner, I. Carol, M.D. Adley, and S.A. Akers. Microplane model M4 for concrete: I. Formulation with work conjugate deviatoric stress. J. Engrg. Mech., ASCE 126(9), (2000), 944-953.
[16] J. Lee, and G.L. Fenves. Plastic-damage model for cyclic loading of concrete structures. J. Struct. Engrg. ASCE, 124(8), (1998), 892-900.
[17] J. Lubliner, J. Oliver, S. Oller, and E. Onate. A plastic-damage model for concrete. Int. J. Solids Structures, 25(3) (1989), 299-326.
[18] M. Jiŕasek and Th., Zimmermann Rotating crack model transition to scalar damage. J. Engrg. Mech., ASCE. 124, (1998), 277-284.
[19] ABAQUS 6.11 documentation SIMULIA, Providence, RI.
[20] M. Jiŕasek and M. Bauer Numerical aspect of the crack band approach. Comp. Struct. Vol.110-111, (2012), 60-78.