Fatigue Crack Propagation in Plain Concrete using Incomplete Self-similarity

S. Ray*, J. M. Chandra Kishen
Indian Institute of Science, Bangalore, India

Abstract

In this work, a fatigue crack propagation law for plain concrete is developed using the concepts of dimensional analysis and self-similarity. This model considers the effect of loading frequency, grade of concrete, fracture parameters, load history and size effect. The dependency of the fatigue parameter on characteristic length, and frequency of loading is reexamined on the basis of theoretical arguments.

[1] Introduction

It is well understood that fatigue accounts for majority of the material failures and has been studied extensively for metallic structural components, where it causes irreversible material damage [1]. But in case of concrete, the fatigue mechanism is different from that in metals due to the heterogeneity of the material.
Fracture of concrete is characterized by the presence of fracture process zone (FPZ) at the crack tip (Fig. 1). The FPZ is a zone in which the cement matrix is intensively cracked. Along the FPZ, there is discontinuity in displacements but not in the stresses. The stresses are themselves a function of the crack opening displacement (COD). At the tip of the FPZ, tensile stress is equal to tensile strength f_t, and it gradually reduces to zero at the tip of the true crack.
It is assumed that under fatigue loading, the resulting damage, that is, the decrease in load carrying capacity and stiffness, occurs primarily in the FPZ not in the undamaged material [2]. In most of the nonlinear material models, it is assumed that only the FPZ is responsible for the variation of material properties during cyclic loading. If the fracture process zone exhibits greater sensitivity to fatigue loading than the surrounding material, then the fatigue behaviour can be considered to be dependent on loading history [8]. Thus, loading history is of paramount importance in understanding the fatigue behaviour of concrete.
In this work, a fatigue crack propagation model is proposed based on theoretical arguments, considering all the parameters which affect the fatigue behaviour of concrete. The dependency of the parameter C on characteristic length (l_{ch}) and frequency of fatigue loading is reexamined on the basis of theoretical arguments. Dimensional analysis and self-similarity concepts are applied to the fatigue propagation model to verify this dependence.

* Corresponding author
E-mail address: sonalisa@civil.iisc.ernet.in

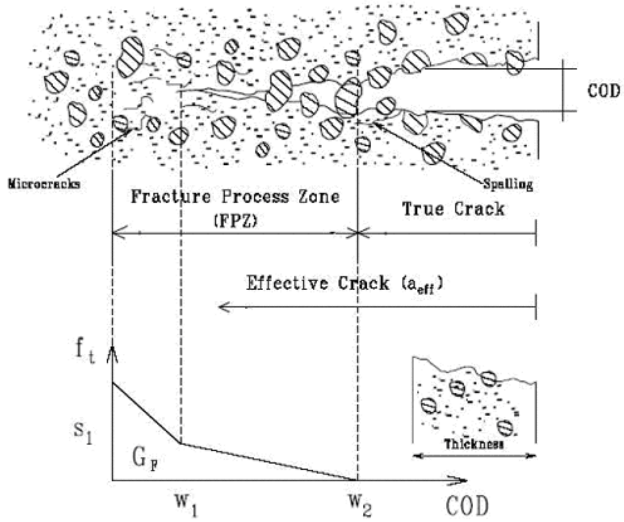

Figure 1: Fracture process zone (Slowik et al., 1996)

.

[2] Literature review

A number of empirical laws characterizing fatigue crack growth have been proposed for metals. Paris law [1] is the most commonly used fatigue crack propagation law.

$$\frac{da}{dN} = C\left(\Delta K_I\right)^m \tag{1}$$

where a is crack length, N the number of cycles, ΔK_I the stress intensity factor range, and C and m are material constants. Bazant and Xu [6] have proposed a size adjusted Paris law, based on size effect as

$$\frac{da}{dN} = C\left(\frac{\Delta K}{K_{IC}}\right)^n \tag{2}$$

where,

$$K_{IC} = K_{If}\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{2}};\ \beta = \frac{d}{d_0} \tag{3}$$

K_{If} is the fracture toughness of an infinitely large structure, d is the characteristic dimension of the structure and d_0 is an empirical constant. This size adjusted Paris law has been validated for constant amplitude loading and for one particular load range only. A number of other empirical laws based on the variation of Paris law have been proposed. Most commonly used are Walker law [3] and Foreman law [2]. Walker law is described by

$$\frac{da}{dN} = C\left(K_{\mathrm{Im}ax}^{m}\right)\Delta K_I^n \tag{4}$$

where $K_{\mathrm{Im}ax}$ is the stress intensity factor corresponding to the upper fatigue load limit and *C*, *n* and *m* are material constants. Foreman's law defined by

$$\frac{da}{dN} = C\,\frac{K_{\mathrm{Im}ax}\;\Delta K_I^m}{K_{IC} - K_{\mathrm{Im}ax}} \tag{5}$$

accounts for the crack growth rate when the maximum stress intensity factor approaches the fracture toughness. But none of the above laws account for the effect of loading history. Slowik et al. [8] have proposed a simplified linear elastic model based on the variation of Paris law which accounts for (1) effect of loading history (2) acceleration effect of spikes and (3) size effect. The proposed model is described by

$$\frac{da}{dN} = C\,\frac{K_{Imax}^{m}\Delta K_I^n}{\left(K_{IC} - K_{I\sup}\right)^p} + F\left(a,\ \Delta\sigma\right) \tag{6}$$

where $K_{I\sup}$ is the maximum stress intensity factor ever reached by the structure in its past loading history; K_{IC} is the fracture toughness; K_{Imax} is the maximum stress intensity factor in a cycle; N is the number of cycles; $F\left(a,\ \Delta\sigma\right)$ is a function that takes into account the sudden overload on the crack propagation and *m*, *n*, *p* are constants for all structural sizes. These constant coefficients are determined through an optimization process using experimental data and the values obtained are 2.0, 1.1, and 0.7 respectively. The parameter *C* is the crack growth rate per fatigue load cycle and its value was found to be 9.5 x 10^{-3} and 3.2 x 10^{-2} mm/cycle for large and small specimens, respectively. The closed form expression for the parameter *C* is a linear relationship between *C*, and the ratio of ligament length (*L*) to characteristic length (l_{ck}) given by,

$$C = \left(-2 + 25\frac{L}{l_{ch}}\right)10^{-3}\ \text{mm/cycle} \tag{7}$$

This equation does not account for the frequency of fatigue loading. Sain and Chandra Kishen [14] have proposed a modified closed form equation by considering the frequency of applied loading (*f*) as

$$Cf = -0.0193\left(\frac{L}{l_{ch}}\right)^2 + 0.0809\left(\frac{L}{l_{ch}}\right) + 0.0209 \text{ mm/sec} \tag{8}$$

This fatigue model developed for plain concrete is established through a regression analysis using experimental data. Spagnoli [11] derived a crack size dependent Paris law based on both similarity and fractal concepts. Ritchie [12] examined the size dependence of the Paris law using the concepts of dimensional analysis and self-similarity. Carpinteri and Paggi [13] have examined the existence of a correlation between the parameters *C* and *m* of the Paris law using self-similarity.

[3] Scaling laws

Scaling laws or power-laws which describe the power-law relationship between different quantities give the evidence of a very important property of self-similarity, wherein a phenomenon reproduces it self on different time and space scales. In construction of an analytical model, it is impossible to take into account all the factors which influence the phenomenon. So, every model is based on certain idealization of the phenomenon. In constructing the idealizations, the phenomenon under study should be considered at intermediate times and distances. Therefore every mathematical model is based on intermediate asymptotic. In fact self-similar solutions not only describes the behaviour of the physical systems under some special conditions but also describes the 'intermediate asymptotic' behaviour of the solution to broader classes of problems i.e. the behavior in the regions where these solutions have ceased to depend on the details of the initial conditions and boundary conditions [7,10]. In this work the phenomenon of fatigue crack propagation on the basis of similarity approach is considered.

[4] Dimensional analysis and self-similarity

In any physical problem, we try to determine the relationship among the physical quantities involved. Let us consider, there exists a relationship between a quantity a which is to be determined from experiments (governed parameter), and a set of quantities which are under experimental control (governing parameters), that can be written as

$$a = f\left(a_1,.......a_k, a_{k+1},....a_n\right) \tag{9}$$

where $\left(a_1......a_k\right)$ have independent physical dimensions, i.e. none of these quantities has a dimension that can be represented in terms of a product of powers of dimensions of the remaining quantities and $\left(a_{k+1}..........a_n\right)$ can be expressed as the product of powers of the dimensions of the parameters $(a_1......a_k)$. This would mean that the dimension of the governed parameter a is determined by the dimensions of $(a_1......a_k)$. Thereby, a_{k+1} can be written as $a_{k+1}/a_1^p......a_k^r$ to make it dimensionless. Introducing the dimensionless parameters as:

$$\Pi_i = \frac{a_{k+i}}{a_1^{p_{k+i}}......a_k^{r_{k+i}}} \quad (10a)$$

$$\prod = \frac{a}{a_1^p........a_k^r} \quad (10b)$$

$$\prod = \Phi\left(\prod_1,........\prod_{n-k}\right) \quad (10c)$$

where Φ is a function of non-dimensional terms

$$a = f\left(a_1,...a_k, a_{k+1},....a_n\right) = a_1^p......a_k^r \Phi\left(\frac{a_{k+1}}{a_1^{p_k+1}........a_k^{r_k+1}},,,,\frac{a_n}{a_1^{p_n}........a_k^{r_k+1}}\right) \quad (11)$$

Applying Buckingham $\prod$ theorem Φ turns out to be a function of (*n*-*k*) variables only. The quantities $\prod_1,........\prod_{n-k}$ are called similarity parameters, and the physical phenomenon is termed similar if the dimensionless parameters $\prod_1,........\prod_{n-k}$ are identical. We shall now discuss two important terms associated with dimensional analysis. (1) Self-similarity of first kind (2) Self-similarity of second kind. Let us consider the parameter a_n. This parameter is considered as non-essential if the corresponding dimensionless parameter $\prod_n$ is too large or too small (tend to zero or infinity), giving rise to a finite non-zero value of the function Φ with the other similarity parameters remaining constant. The number of arguments can now be reduced by one and we can write

$$\prod = \Phi_1\left(\prod_1,............\prod_{n-1}\right) \quad (12)$$

where Φ_1 is the limit of the function Φ as $\prod_n \to 0$ or $\prod_n \to \infty$. This is called complete self-similarity or self-similarity of first kind. On the other hand for $\prod_n \to 0$ or $\prod_n \to \infty$, if Φ tends to zero or infinity, then the quantity $\prod_n$ becomes essential, no matter how large or small it becomes. However in some cases, the limit of the function Φ tends to zero or infinity, but the function Φ has power type asymptotic representation which can be written as,

$$\Phi \cong \prod_n^\beta \Phi_1\left(\prod_1,............\prod_{n-1}\right) \quad (13)$$

$$\Pi^* = \frac{\Pi}{\Pi_n^{\beta}} \Phi_1 \left(\Pi_1, ...\Pi_{n-1} \right) \tag{14}$$

where the constant β and the non-dimensional parameter Φ_1 cannot be obtained from the dimensional analysis alone. This is the case of incomplete self-similarity or self-similarity of second kind.

[5] Fatigue crack propagation for plain concrete and self-similarity

The fatigue crack propagation phenomenon is generally analyzed in the medium amplitude range and in this range the phenomenon is characterized by an 'intermediate asymptotic' nature [5]. Assuming the crack growth rate $\frac{da}{dN}$ as the parameter to be determined in the phenomenon, we can write the following functional dependence.

$$\frac{da}{dN} = \Phi\left(K_{\mathrm{I}max},\ \Delta K_I,\ K_{IC},\ K_{I\sup},\ a,\ l_{ch},\ D,\ \sigma_t,\ \omega,\ t,\ 1-R \right) \tag{15}$$

where the governing variables are summarized in Table1.
Considering K_{IC} , σ_t and ω as independent variables, dimensional analysis gives

$$\frac{da}{dN} = \left(\frac{K_{IC}}{\sigma_t} \right)^2 \Phi\left(\frac{K_{\mathrm{I}max}}{K_{IC}}, \frac{\Delta K_I}{K_{IC}}, \frac{\Delta K_{I\sup}}{K_{IC}},\ \frac{\sigma_t}{K_{IC}}\sqrt{a},\ \frac{\sigma_t}{K_{IC}}\sqrt{l_{ch}},\ \frac{\sigma_t}{K_{IC}}\sqrt{D},\ \omega t,\ 1-R \right) \tag{16}$$

Table 1: Governing Variables of the Fatigue Crack Growth Phenomenon in Plain Concrete

Variables	Definitions	Dimensions
$K_{\mathrm{I}max}$	Maximum stress intensity factor (S.I.F) in a cycle	$FL^{-3/2}$
ΔK_I	Stress intensity factor range	$FL^{-3/2}$
K_{IC}	Fracture toughness	$FL^{-3/2}$
$K_{\mathrm{I}\sup}$	Max S.I.F ever reached in the past loading history	$FL^{-3/2}$
a	Initial crack length	L
l_{ch}	Characteristic length	L
D	Structural size	L
σ_t	Tensile strength	FL^{-2}
ω	loading frequency	T^{-1}
t	Time	T
R	Loading ratio	-

Here, the dimensionless quantities are

$$\Pi_1 = \frac{K_{\mathrm{Im}ax}}{K_{IC}}, \;\; \Pi_2 = \frac{\Delta K_I}{K_{IC}}, \;\; \Pi_3 = \frac{\Delta K_{I\sup}}{K_{IC}}, \;\; \Pi_4 = \frac{\sigma_t}{K_{IC}}\sqrt{a} = \sqrt{\frac{a}{\pi\, r_p}}$$

$$\Pi_5 = \frac{\sigma_t}{K_{IC}}\sqrt{l_{ch}} = \sqrt{\frac{l_{ch}}{\pi\, r_p}}, \;\; \Pi_6 = \frac{\sigma_t}{K_{IC}}\sqrt{D}, \;\; \Pi_7 = \omega\, t, \;\; \Pi_8 = 1 - R$$

Now, we need to analyze whether the number of arguments can be reduced further or not. Considering Π_2, it is usually small in the intermediate range of fatigue crack growth and assuming incomplete self-similarity in Π_2, we can write

$$\frac{da}{dN} = \left(\frac{K_{IC}}{\sigma_t}\right)^2 \left(\frac{\Delta K_I}{K_{IC}}\right)^{\beta_1} \Phi\left(\Pi_1,\ \Pi_3,\ \Pi_4,\ \Pi_5,\ \Pi_6,\ \Pi_7,\ \Pi_8\right) \tag{17}$$

$$= \left(K_{IC}\right)^{2-\beta_1} \sigma_t^{-2} \left(\Delta K_I\right)^{\beta_1} \Phi_1\left(\Pi_1,\ \Pi_3,\ \Pi_4,\ \Pi_5,\ \Pi_6,\ \Pi_7,\ \Pi_8\right)$$

The non-dimensional parameter Π_5 may be assumed to be small since the characteristic length is relatively smaller than the inelastic zone size (r_p). A complete self-similarity in Π_5 would imply that fatigue crack growth is independent of l_{ch}. But from experimental findings [8] it is known that $\frac{da}{dN}$ is dependent on l_{ch}. Hence incomplete self-similarity for Π_5 leads to

$$\frac{da}{dN} = \left(\frac{K_{IC}}{\sigma_t}\right)^2 \left(\frac{\Delta K_I}{K_{IC}}\right)^{\beta_1} \left(\frac{\sigma_t}{K_{IC}}\sqrt{l_{ch}}\right)^{\beta_2} \Phi_3\left(\Pi_1,\ \Pi_3,\ \Pi_4,\ \Pi_6,\ \Pi_7,\ \Pi_8\right) \tag{18a}$$

$$= K_{IC}^{\;2-\beta_1-\beta_2}\ \sigma_t^{\beta_2-2}\ \left(\Delta K_I\right)^{\beta_1}\ \left(l_{ch}\right)^{\beta_2/2}\ \Phi_2\left(\Pi_1,\ \Pi_3,\ \Pi_4,\ \Pi_6,\ \Pi_7,\ \Pi_8\right) \tag{18b}$$

The crack size may be relatively smaller than the inelastic zone size r_p. Therefore, Π_4 can be considered to be small, but experimental results [11] have shown the dependence of crack growth rate on a. Hence, incomplete self-similarity gives

$$\frac{da}{dN} = \left(\frac{K_{IC}}{\sigma_t}\right)^2 \left(\frac{\Delta K_I}{K_{IC}}\right)^{\beta_1} \left(\frac{\sigma_t}{K_{IC}}\sqrt{l_{ch}}\right)^{\beta_2} \left(\frac{\sigma_t}{K_{IC}}\sqrt{a}\right)^{\beta_3} \Phi_3\left(\Pi_1,\ \Pi_3,\ \Pi_6,\ \Pi_7,\ \Pi_8\right) \tag{19a}$$

$$= \left(K_{IC}\right)^{2-\beta_1-\beta_2-\beta_3} \; \sigma_t^{\beta_3+\beta_2-2} \; \left(\Delta K_I\right)^{\beta_1} \; \left(l_{ch}\right)^{\beta_2/2} \; \left(a\right)^{\beta_3/2} \Phi_3\left(\Pi_1,\ \Pi_3,\ \Pi_6,\ \Pi_7,\ \Pi_8\right) \qquad (19b)$$

$\left(1-R\right)$ is a small number that lies between 0 to 1. If we consider a complete self-similarity for Π_8, then it would imply that $\frac{da}{dN}$ is independent of R. This behaviour is in contrast with experimental findings [13]. Therefore, considering incomplete self-similarity, Eq. (19b) can be written as,

$$\frac{da}{dN} = \left(\frac{K_{IC}}{\sigma_t}\right)^2 \left(\frac{\Delta K_I}{K_{IC}}\right)^{\beta_1} \left(\frac{\sigma_t}{K_{IC}}\sqrt{l_{ch}}\right)^{\beta_2} \left(\frac{\sigma_t}{K_{IC}}\sqrt{a}\right)^{\beta_3} \left(1-R\right)^{\beta_4} \Phi_4\left(\Pi_1,\ \Pi_3,\ \Pi_6,\ \Pi_7\right) \qquad (20a)$$

The above equation could be re-written as

$$\frac{da}{dN} = \left(K_{IC}\right)^{2-\beta_1-\beta_2-\beta_3} \; \left(1-R\right)^{\beta_4} \; \left(\Delta K_I\right)^{\beta_1} \; \sigma_t^{\beta_3+\beta_2-2} \; \left(l_{ch}\right)^{\beta_2/2} \; \left(a\right)^{\beta_3/2} \Phi_4\left(\Pi_1,\ \Pi_3,\ \Pi_6,\ \Pi_7\right) \qquad (20b)$$

For the particular case of fatigue crack propagation in concrete defined by Eq. (6), ignoring the effect of overload we can reduce the above model as follows:

1) Writing $K_{I\sup} = sK_{IC}$ where s lies within the limit $0 < s < 1$, as $K_{\mathrm{I}\sup}$ will never reach K_{IC}.
2) Further, we can write $K_{\mathrm{I}max} = \frac{\Delta K_I}{1-R}$.

Substituting the above in Eq. (6)

$$\frac{da}{dN} = \frac{C}{\left(1-R\right)^m \left(1-s\right)^p} \frac{\left(\Delta K_I\right)^{m+n}}{K_{IC}{}^p} \qquad (21)$$

Comparing Eq. (20b) with Eq. (6), we obtain

$$C = \sigma_t^{\beta_3+\beta_2-2} \; a^{\beta_3/2} \; l_{ch}^{\beta_2/2} \; \Phi_4\left(\Pi_1,\ \Pi_3,\ \Pi_6,\ \Pi_7\right) \qquad (22)$$

Here, $\beta_1,\ \beta_2,\ \beta_3,\ \beta_4$ are functions of $\Pi_1,\ \Pi_3,\ \Pi_6,\ \Pi_7$. From Eq. (22), it is observed that the parameter C is dependent on material properties like tensile strength σ_t, fracture toughness K_{IC}, characteristic length l_{ch}, crack length a, and the remaining non-dimensional parameters which also include loading frequency and structural size. This is also observed experimentally by Slowik et al. [8].

[6] Conclusions

In this study, the analytical form of the fatigue crack propagation law is proposed using theoretical arguments. The form of the unknown fatigue parameters *C*, *m*, *n*, and *p* is obtained. From this study it may be concluded that the parameter *C* is a function of characteristic length, loading frequency and structural size. It is observed that *C* is also a function of crack-size. This dependence of *C* on crack size needs to be verified using experimental results. More work is required to understand the physical meaning of the different parameters involved in the proposed fatigue law. Further, closed form expressions for these parameters need to be obtained using experimental data.

[7] References

[1] P. Paris, F. Erdogan, A Critical Analysis of Crack Propagation Laws, J. of Basic Engineering, ASME, 85(3) (1963), 528-534.

[2] R.G. Foreman, V.E. Kearney, Numerical Analysis of Crack Propagation in Cyclic Loaded Structures, J. of Basic Engineering, 94 (1967) 459-464.

[3] E.K. Walker, Effect of Stress Ratio during Crack Propagation and Fatigue for 2024-t3 and 7075-t6 Aluminum, Effects of Environment and Complex Load History on Fatigue Life, American Society for Testing and Materials STP 462, (1970).

[4] G.I. Barenblatt, Self similarity: Dimensional Analysis, and Intermediate Asymptotic. PMM U.S.S.R., 44(2) (1980) 267-272.

[5] G.I. Barenblatt, L.R. Botvina, Incomplete Self-similarity of Fatigue in the Liner Range of Crack Growth., Fatigue Fract. Eng. Mater. Struct.,3 (1980), 193-202.

[6] Z. P. Bazant, K. Xu Size effect in fatigue fracture of concrete. ACI Materials Journal, 88(4) (1991) 427-437.

[7] G.I. Barenblatt, Scaling, Self-similarity and Intermediate Asymptotic. Cambridge University Press, Cambridge, 1996.

[8] V. Slowik, G. A. Plizzari and V. E. Saouma Fracture of Concrete under Variable Amplitude Loading. ACI Materials Journal, 93(3) (1996) 272-283.

[9] J. Zhang, H. Stang and V. C. Li, Fatigue Life Prediction of Fiber Reinforced Concrete under Flexural load. Int. J. of Fatigue, 21, (1999) 1033-1049.

[10] G.I. Barenblatt , Scaling, Cambridge University Press, Cambridge, 2004.

[11] A. Spagnoli, Self-similarity and Fractals in the Paris range of Fatigue Crack Growth, Mechanics of Materials, 37 (2004) 519-529.

[12] R.O. Ritchie, Incomplete Self-similarity and Fatigue Crack Growth. Int. J. of Fracture, 132 (2005) 197-203.

[13] A. Carpinteri, M. Paggi, Self-similarity and Crack Growth Instability in the Correlation between the Paris Constants, Engg. Fracture Mechanics, 74 (2007) 1041-1053.

[14] T. Sain, J. M. Chandra Kishen, Prediction of Fatigue Strength in Plain and Reinforced Concrete Beams, ACI Structural Journal, 104(5), (2007) 621- 628.