Effect of interaction between collinear cracks on the strength distribution of brittle materials

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ABSTRACT

Weibull theory neglects interaction between the defects and between defects and stress field, and accordingly obtains that the strength of a material follows a Weibull distribution whose modulus is a material constant, and whose mean value only is affected by the geometry and loading condition. For the limit case of a cracked structure, this corresponds to a paradoxical zero mean strenght. A full account of interaction requires direct Montecarlo simulations (each elasticity problem being solved by an efficient dual boundary element formulation) and shows that mean value and scatter deviate from the theoretical ones. A simple geometry has been considered where an infinite plate uniform remote tension σ_∞, containing N collinear cracks with statistical distribution of sizes and spacings.

1 Introduction

The strenght of brittle materials was studied by Weibull heuristically as one of the applications of his known distribution [1], [2]. Much later, Freudenthal [3] showed then that this distribution can be explained with the existence of *flaws*, neglecting mutual interaction in the stress fields surrounding each flaw, and applying Griffith equation which specifies the critical length $2a$ of an elliptical crack in terms of the stress intensity σ

$$\sigma\sqrt{a} = k = \text{const} \tag{1}$$

where the constant k depends on the Young's modulus E, Poisson's modulus ν and the rate of work Γ_c per unit area of crack.

In particular, according to *Weakest Link Theory* (WLT), the fracture of a specimen is identified with the unstable propagation of the most "critical" crack (the largest in a uniform stress field). Peirce [4] was the first to formulate WLT and recognized the close relation of this model to the theory of extremes values, stating that the distribution of smallest values tends in the limit of large number of samples to be one of the two physically significant *asymptotic distributions*, regardless of the initial population (Gumbel, [5]). The advantage of this approach results from the fact that only two kinds of physically significant distribution functions of extreme values exist: one function represents the extremes of unlimited initial populations, described by functions that converge towards zero for $|x| \to \infty$ at least as fast as the exponential function $\exp(-x)$; the other function represents the extremes of initial populations (called Cauchy type distributions) that are limited for $x = 0$ and converge towards zero for $|x| \to \infty$ as fast as an inverse power law x^{-m}. For example, for a Cauchy type distributions of crack size a, the function distribution of the largest cracks was derived by Frechet [6]

$$F_a(a) = \exp\left[-\left(\frac{a}{u}\right)^{-\alpha}\right] \tag{2}$$

with u characteristic size. By using Griffith's equation, eqt. (1) can be converted into a function of the strength distribution

$$F_\sigma(\sigma) = 1 - \exp\left[-\left(\frac{k^2}{\sigma^2 u}\right)^{-\alpha}\right] = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^{2\alpha}\right] \tag{3}$$

with $\sigma_0 = k/\sqrt{u}$. Such distribution coincides with the two-parameter Weibull distribution with modulus $m = 2\alpha$. Considering also the effect of the volume, in terms of probability of survival $R(V) = 1 - F(V)$,

$$R(\sigma, V) = \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m \frac{V}{V_0}\right] = \exp\left[-n_c(\sigma)\, V\right] \tag{4}$$

The function $n_c(\sigma) = \frac{1}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m$ is a stress dependent "risk of rupture", where V_0 is representative of the material volume, σ_0 is a scale parameter and m is the shape parameter (known as Weibull modulus) which is a measure of flaw size scatter. The volume component is easily derived as it is the only one satisfying the WLT condition (see Freudenthal [3])

$$R(\sigma, nV) = [R(\sigma, V)]^n, \tag{5}$$

Notice that this condition could be satisfied with $n_c(\sigma)$ *any* function of the stress σ. The Weibull distribution simply corresponds to the case

$$n_c(\sigma) = \left(\frac{\sigma}{\sigma_0}\right)^m \qquad \text{or} \qquad n_c(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0 - \sigma_u}\right)^m \tag{6}$$

where the latter case is for three parameter distribution. For non uniform stress field, we immediately derive the product as an integral,

$$R = \exp\left[-\frac{1}{V_0}\int_V \left(\frac{\sigma}{\sigma_0}\right)^m dV\right] \tag{7}$$

However, writing WLT in this form is purely speculative as it corresponds to effectively assuming infinitesimal volumes of material non interacting with each other — a distribution of cracks cannot be collapsed to a single material point! Therefore, the larger the stress gradient, or the larger the cracks with respect to their distance, the larger the deviation we expect from the simple Weibull case.

In order to see the effect of gradient, let's consider the case of a macroscopic crack or sharp notch. Using the Williams' asymptotic stress field for a notch $-\alpha < \theta < \alpha$,

$$\sigma(r, \theta) = Ar^{-p} g(\theta) \tag{8}$$

where A is an arbitrary constant, p is a dimensionless exponent and g is some function, we obtain

$$R = \exp\left[-\frac{1}{V_0}\int_{-\alpha}^{\alpha} \left(\frac{Ag(\theta)}{\sigma_0}\right)^m d\theta \int_0^{\infty} r^{1-mp} dr\right] \tag{9}$$

Now,

$$\int_{\rho_1}^{\rho_2} r^{1-mp} dr = \frac{\rho_2^{2-mp} - \rho_1^{2-mp}}{(2-mp)} \tag{10}$$

There are two possibilities. If $mp > 2$, the integral will be unbounded when $\rho_1 \to 0$, whereas if $mp < 2$, it will be unbounded when $\rho_2 \to \infty$. For $mp = 2$ it is unbounded at both limits. Thus, for all values of m, p the probability of survival is zero, but for different reasons. For $mp > 2$, failure occurs because of the high stress in the notch, whereas for $mp < 2$ it occurs because of the unbounded volume of lightly stressed material away from the notch. In the latter case, bounded results would be obtained by considering the actual stress field in the finite body.

However, the main conclusion is that the mean value of strenght is likely to be zero, whereas we expect it to be finite (as after all the macroscopic crack is no qualitatively different from the distribution of cracks giving the Weibull statistics). Also, there is no particular reason to expect the scatter to be given by the Weibull modulus in the case of pure tension. In fact, Weibull modulus depends on loading and geometrical factors, and is not a material constant as noticed already experimentally by various authors (Milella and Bonora, [7]). In order to attack the problem of interaction, we start from a simple case, generating distributions of collinear cracks in a large plate under uniform tension, and finding the strength distributions by an efficient dual boundary element method (DBEM).

2 Formulation

The geometry of the problem is shown in *Figure 1*. The model is similar to that considered by Su-Lin Zhang et al. [8]. In particular, N cracks are considered with arbitrary lengths (normally distributed) and equal ligament sizes.

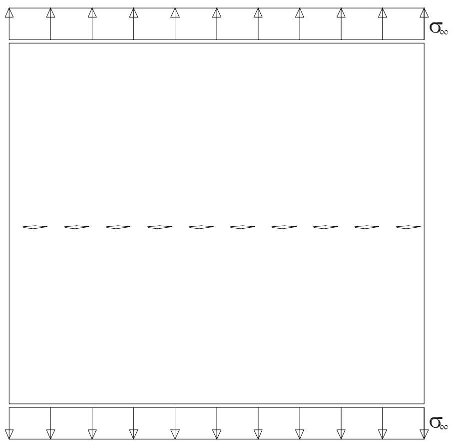

Figure 1: Infinite plate with N collinear cracks,
loaded on the edges by uniform remote tension σ_∞.

The ultimate stress σ_r has been evaluated as

$$\sigma_r = \sigma_\infty \frac{K_{IC}}{K_I} \tag{11}$$

where K_I and K_{IC} are, respectively, the stress and the critical stress intensity factor, and the results has been interpreted by the Weibull distribution.

The equation that relates the Weibull modulus m to the number of experimental data (Smart et al. [9]) is

$$\frac{m_{sd}}{m_0} = \frac{1}{\sqrt{N_g}} \tag{12}$$

where N_g is the number of data and m_0 and m_{sd} are the expected value and standard deviation of the Weibull modulus. Eq. (12) shows as the scatter of the modulus m decreases with the number of data N_g and the larger is m the larger has to be N_g.

Figure 2a and *2b* show the values of Weibull modulus obtained, respectively, by a three and two parameters regression on samples with $N_g = 25 \div 500$. For each specimen, the ligament size is distributed with normal law with mean value $c_0/a_0 = 0.5$, and standard deviation $c_{sd} = 0.2$.

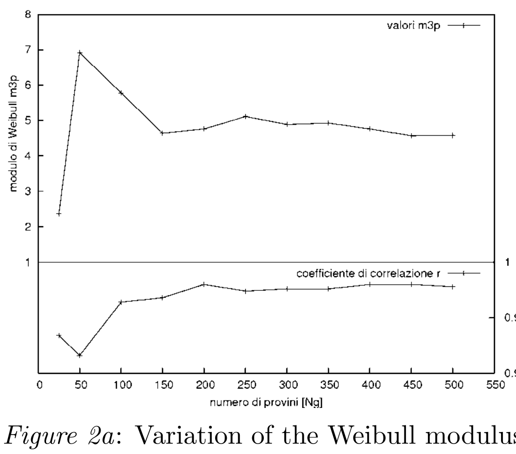

Figure 2a: Variation of the Weibull modulus
with the number of specimens
(three parameters distribution).

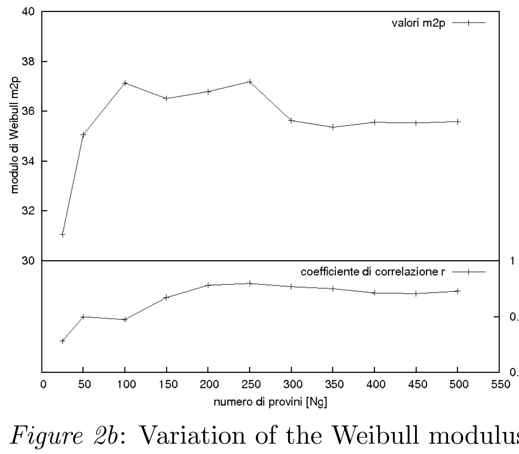

Figure 2b: Variation of the Weibull modulus
with the number of specimens
(two parameters distribution).

The number N_g of specimens needed to identify with enough reliability the correct distribution can be very large. However, we set $N_g = 100$ as the estimated error is around 4%. Further, the number of cracks of the specimens is bounded to $N_c = 50$, finding a trade-off between reliability and computational costs.

In *Figure 3* is shown the variation of the Weibull modulus m with the number of cracks N_c for specimen.

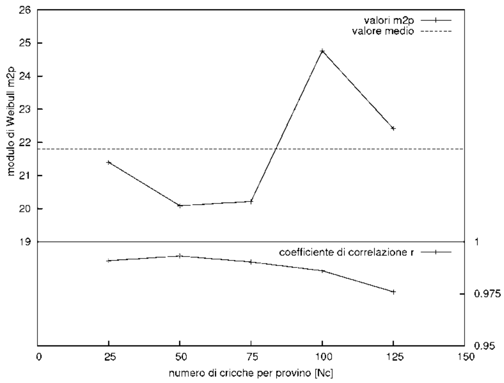

Figure 3: Variation of the Weibull modulus m with the number of cracks N_c for specimen.

The values of m are in the range between 20.1 and 24.8. However, by generating samples of specimens with the same average and standard deviation of cracks length, the statistical regression of the results yields different values of m, as shown in *Table 1*.

Sample	Weibull modulus m_{2p}	correlation factor r
1	37.1304	0.97367
2	34.8360	0.99458
3	31.8717	0.98224
4	33.8519	0.97897
5	33.9206	0.99280
6	32.8556	0.99162
7	38.3108	0.98981
8	32.3828	0.98344

Table 1: Weibull modulus m_{2p} for samples of specimens with the same average and standard deviation ($a_0/c_0 = 1$; $a_{ds} = 0.3$).

Notice that the results show a variation of m of $\pm 6.7\%$.

2.1 Cracks distributed with normal law

In this section a first investigation on the interaction effects between the cracks is shown. In *Figure 1*, we considered 50 collinear cracks with identical ligament size c_0. A normal distribution was considered for the cracks length $2a_i$ with fixed mean a_0 and standard deviation a_{sd}. Thirty-six samples, each with $N_g = 100$ specimens, were generated with the following values of average a_0 and standard deviation a_{sd}

$$\begin{aligned} a_0/c_0 &\in [0.05;\ 0.1;\ 0.5;\ 1;\ 2;\ 4] \\ a_{ds}/c_0 &\in [0.1;\ 0.2;\ 0.3;\ 0.5;\ 1;\ 5] \end{aligned}$$

The normal distributions with the lower values of a_0/c_0 and the larger of a_{sd}/c_0, were truncated to avoid the possibility of obtaining negative values of the ligament size.

In *Figure 4* is shown the probability failure vs the normalized strength ($\sigma_r = \sigma_0 K_{IC}/K_I$) for $a_0/c_0 = 0.5$ and $a_{ds}/c_0 = 0.3$.

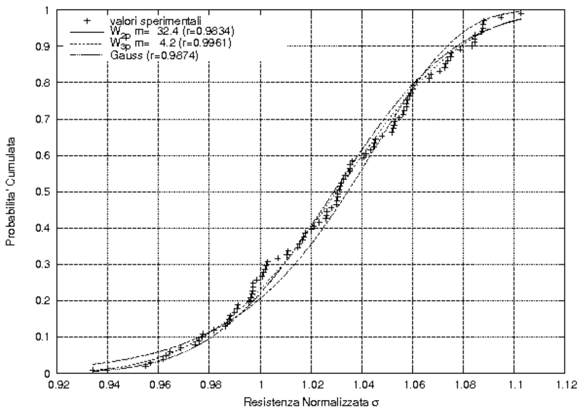

Figure 4: Probability failure vs normalized strengt
($a_0/c_0 = 0.5$ and $a_{ds}/c_0 = 0.3$).

The numerical results are better fitted by the three parameters Weibull distribution than the normal and two parameters ones. However, in the central zone, where is concentrated the larger number of data, the two parameters distribution has a good correlation with the numerical results, and the simpler two parameters distribution is used in the following.

Figure 5a and *5b* show, respectively, the variation of the Weibull modulus m_{2p} with the mean value a_0/c_0 and the standard deviation a_{sd}/c_0.

Notice that increasing the average value of the cracks length the Weibull modulus increases, stating that when the cracks length grows, the scatter of the material strength decreases. The modulus m_{2p} grows more quickly when the standard deviation decreases according to the fact that for standard deviation $a_{sd}/c_0 = 0$, the strength value becomes deterministic. Such behavior is underlined in *Figure 5b*, where the scatter of the results increases (lower values of m_{2p}) with the standard deviation.

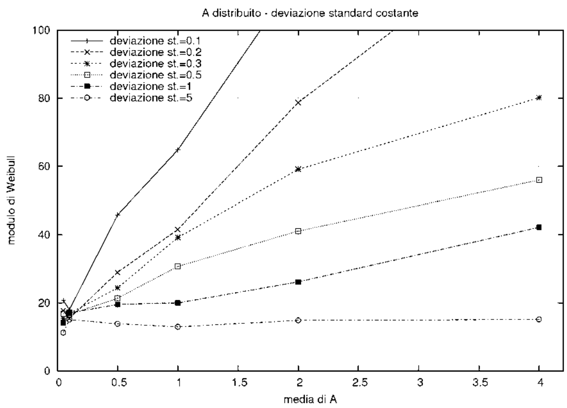

Figure 5a: Variation of the Weibull modulus m_{2p} with tha average a_0/c_0.

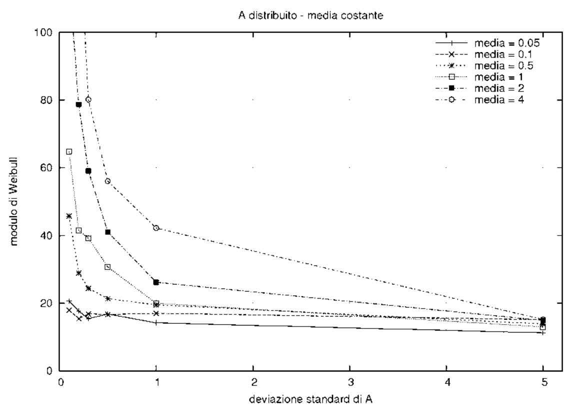

Figure 5a: Variation of the Weibull modulus m_{2p} with tha standard deviation a_{sd}/c_0.

3 Conclusions

The usual definition of the Weibull modulus as a material constants implicitely assumes that there is no interaction between cracks, and that the stress field is sufficiently slowly varying, for the crack stress intensity factors to depend only on the local value of stress. Although the Weibull distribution is very commonly found in experiments (and indeed also in our numerical results), it does not necessarily have the parameters expected from the statistics of extremes as applied to the distribution of cracks in the specimen. On the contrary, for a given distribution of cracks, the stress field functional form and interaction of defects may affect both mean and modulus.

Acknowledgements

The authors wish to thank the Ministero dell'Istruzione dell'Università e della Ricerca for the financing PROMOMAT - Decree n° 947/Ric. 8/7/ 2002 on the Special Fund for the Strategic Research Development – FIST Law 449/97 Art.51, Paragraph 9.

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