Elastic study of a cracked layer bonded to a viscoelastic substrate

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Abstract

The effect of a viscoelastic substrate on an elastic cracked layer under an in-plane concentrated load is solved and discussed in this study. Based on a correspondence principle, the viscoelastic solution is directly obtained from the corresponding elastic one. The elastic solution in an anisotropic trimaterial is solved as a rapidly convergent series in terms of complex potentials via the successive iterations of the alternating technique in order to satisfy the continuity condition along the interfaces between dissimilar media. This trimaterial solution is then applied to a problem of a thin layer bonded to a half-plane substrate. Using the standard solid model to formulate the viscoelastic constitutive equation, the real time stress intensity factors can be directly obtained by performing the numerical calculations. The results obtained in this paper are useful in studying the problem with bone defects where a crack is assumed to exist in an elastic body made of the cortical bone that is bonded to a viscoelastic substrate made of the cancellous bone.

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1. Introduction

In this study, we use an alternating technique to solve the in-plane stress intensity factor of a crack lying in an anisotropic elastic thin layer bonded to a viscoelastic substrate. The solution associated with singularities in dissimilar media is derived from that associated with singularities in the corresponding homogeneous medium. With the aid of the dual coordinate transformation, a singular integral equation is solved to obtain the asymptotic solution to a crack with arbitrary orientations. Using the standard solid model to simulate the viscoelastic constitutive equation and applying an inverse Laplace transform by the aid of Mathematica software, the real-time stress intensity factors are determined. Numerical examples of a crack located arbitrarily in a thin layer made of the cortical bone bonded to a viscoelastic substrate made of the cancellous bone are considered and discussed in detail.

2. Stress intensity factors

The Laplace transform of the stress intensity factor for the present problem can be ob tained as

$$\hat{\boldsymbol{K}} = \lim_{r\to 0} \sqrt{2\pi r}\, \hat{\boldsymbol{t}}^{*} \qquad 1$$

where

$$\hat{\boldsymbol{t}}^{*} = 2\Re\left\{\hat{\boldsymbol{L}}_b^{*}\, \hat{\boldsymbol{f}}_b'(\hat{z})\right\} \qquad 2$$

with

$$\hat{\boldsymbol{f}}_b'(\hat{z}) = \sum_{n=1}^{\infty}\left[\hat{\boldsymbol{f}}_n'(\hat{z}) + \hat{\bar{\boldsymbol{V}}}_{ab}\, \hat{\bar{\boldsymbol{f}}}_n'\left(\hat{z} - \hat{\mu}_b h + \hat{\bar{\mu}}_b h\right)\right] \qquad 3$$

in which the recurrence formula for $\hat{f}_n'(\hat{z})$ is

$$\hat{\boldsymbol{f}}_{n+1}'(\hat{z}) = \begin{cases} \hat{\boldsymbol{f}}_0'(\hat{z}) + \hat{\bar{\boldsymbol{V}}}_{cb}\, \hat{\bar{\boldsymbol{f}}}_0'(\hat{z}) & n = 0 \\ \hat{\bar{\boldsymbol{V}}}_{cb}\hat{\boldsymbol{V}}_{ab}\, \hat{\boldsymbol{f}}_n'\left(\hat{z} - \hat{\bar{\mu}}_b h + \hat{\mu}_b h\right) & n = 1, 2, 3, \Lambda \end{cases} \qquad 4$$

Note that the 3×3 matrices $\hat{\boldsymbol{V}}_{ab}$ and $\hat{\boldsymbol{V}}_{cb}$ in Eqs. (3) and (4) are defined as

$$\hat{\boldsymbol{V}}_{ab} = \hat{\bar{\boldsymbol{L}}}_b^{-1}\left(\hat{\bar{\boldsymbol{B}}}_b\right)^{-1}\left(\hat{\boldsymbol{B}}_b\right)\hat{\boldsymbol{L}}_b$$

$$\hat{\boldsymbol{V}}_{cb} = \hat{\bar{\boldsymbol{L}}}_b^{-1}\left(\hat{\boldsymbol{B}}_c + \hat{\bar{\boldsymbol{B}}}_b\right)^{-1}\left(\hat{\boldsymbol{B}}_b - \hat{\boldsymbol{B}}_c\right)\hat{\boldsymbol{L}}_b$$

where L and B are the material constants. Then, the real time stress intensity factor can be directly obtained by taking the inverse Laplace transform of Eq. (1).

3. Numerical results

The main interest here is to analyze the bone defect where a crack is embedded in the cortical bone bonded to the cancellous bone having a viscoelastic property (see Table 1). In the following discussion, the center of the crack with length $a/h = 0.25$ is fixed at $x_{c1} = 0, x_{c2} = 0.5h$ and the concentrated force is located at $x_{D1} = 0, x_{D2} = h$ (see Fig. 1). All the calculated results are, which are presented only at the right crack tip, evaluated with terms up to $n=4$ in Eq. (3). The normalized stress intensity factors K_I and K_{II} versus the evolution of time under a negative horizontal concentrated force ($\boldsymbol{P} = [\text{-}P_1, 0]$) with the dimensionless factor $K_0 = P_1 \big/ 2\sqrt{\pi a}$ are shown in Fig. 2 and Fig. 3, respectively. For a given crack angle, as shown in Fig. 2, the K_I value converges to its long-time value as time elapses. This convergent value is treated as the elastic response. It shows that the K_I value increases with time until it approaches a long-time constant value for the crack angle $\theta = 30^0$, $\theta = 45^0$ and $\theta = 60^0$ while the K_I value almost remains constant as time elapses for the crack angle $\theta = 90^0$. It is clear that the effect of cancellous bone is less significant for the stress intensity factor K_I when a crack is placed parallel or normal to the free surface under a horizontal concentrated force. It also exhibits that, for various crack angles, the maximum K_I value occurs at $\theta = 45^0$ and the minimum K_I value at $\theta = 0^0$. In contrast with the variation of K_I in the case of the horizontal loading, the effect of cancellous bone is more significant for the stress intensity factor K_{II} when a crack is placed parallel ($\theta = 0^0$) or normal ($\theta = 90^0$) to the free surface (see Fig. 3). In addition, the maximum K_{II} value occurs at $\theta = 0^0$ and the minimum occurs at $\theta = 45^0$.

Table 1. Material properties of a thin layer medium and a half space substrate

	E_1 MPa	E_2 MPa	ν_{12}	ν_{23}	G_{12} MPa
Thin layer (Cortical bone)	20×10^3	18.7×10^3	0.24	0.29	4.6×10^3
Substrate (Cancellous bone)	1×10^3	0.8×10^3	0.315	0.35	0.24×10^3
	$\lambda=\eta=1$				

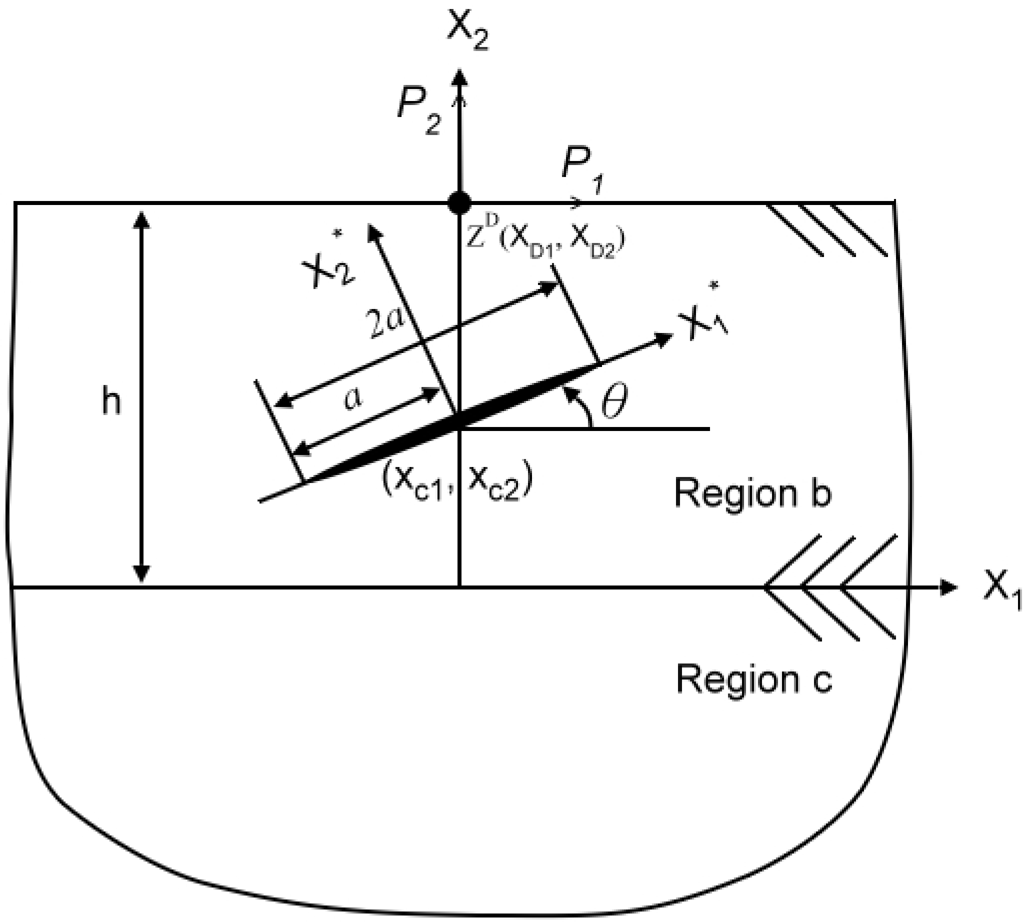

Fig. 1　A crack embedded in the thin layer medium with a half space substrate.

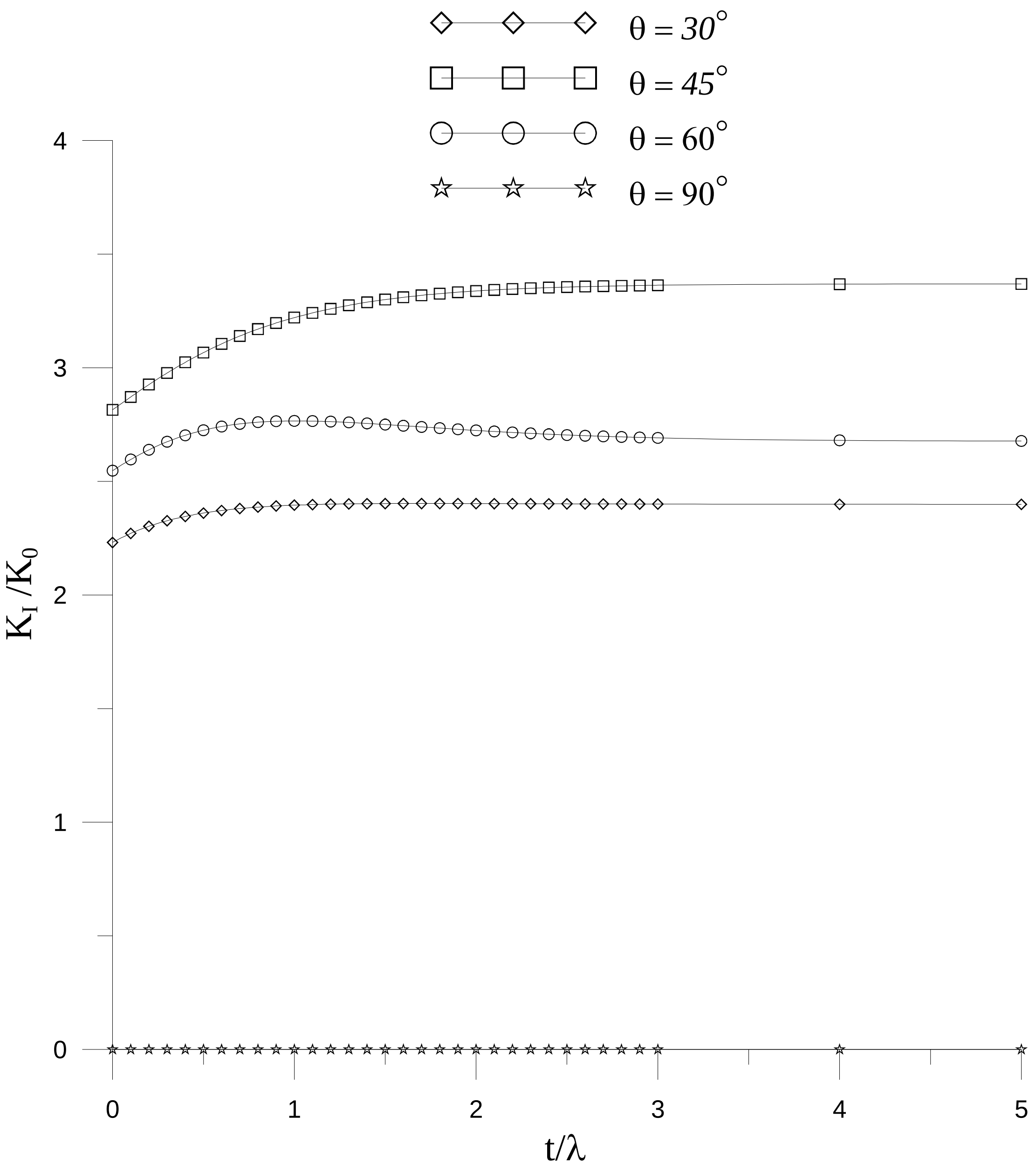

Fig. 2 Mode-I stress intensity factor K_I for various crack orientations subjected to a horizontal concentrated force $\boldsymbol{P} = [-P_1, 0]$ with *a/h=0.25*

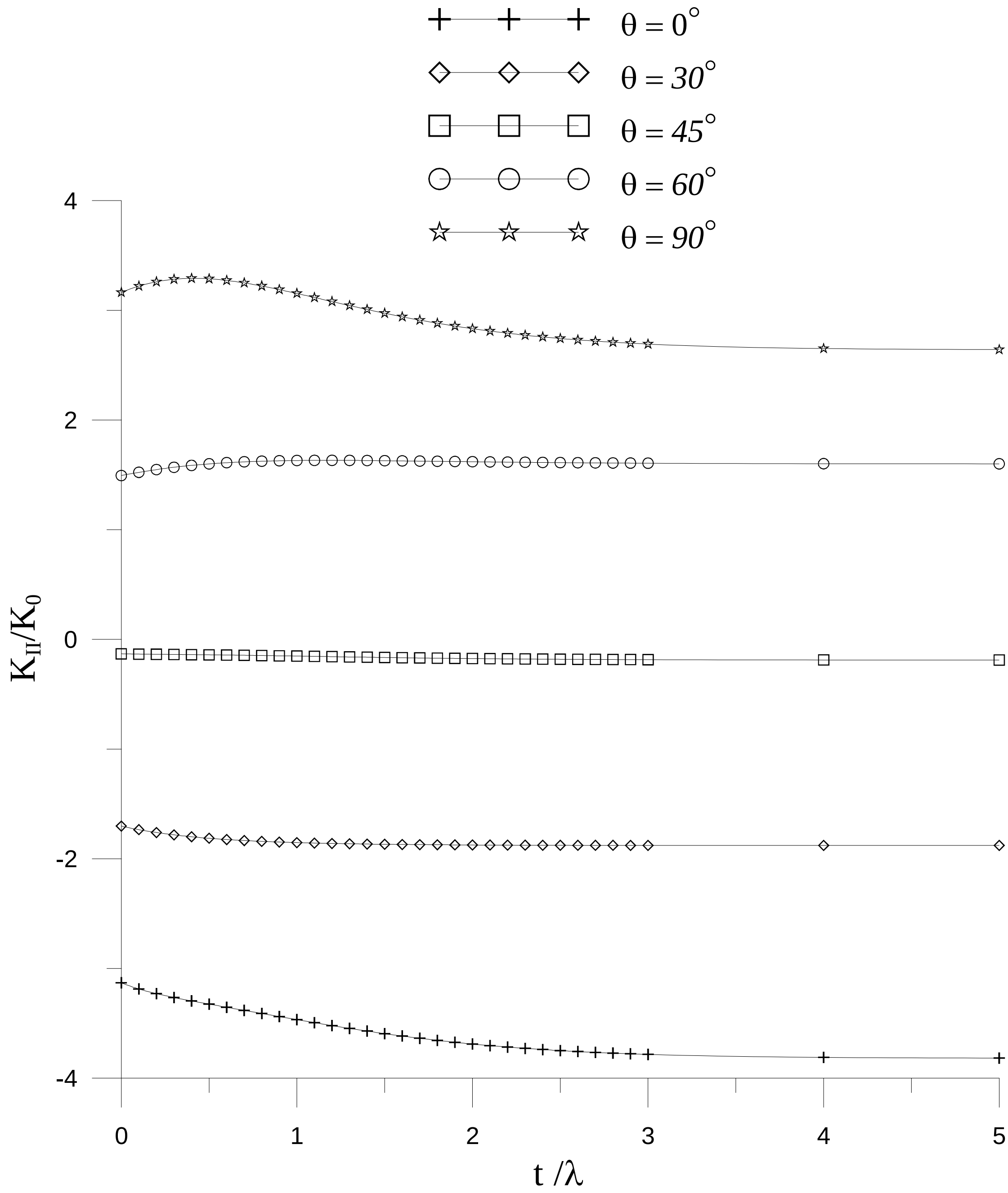

Fig. 3 Mode-II stress intensity factor K_{II} for various crack orientations subjected to a horizontal concentrated force $\boldsymbol{P} = \left[-P_1, 0\right]$ with *a/h=0.25*