A NONLOCAL INTEGRAL APPROACH TO ELASTIC-DAMAGE INTERFACE MODELLING

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ABSTRACT

The paper presents an elastic-damage interface model developed with constitutive relations based on nonlocal concepts. The main motivation for accepting to pay the cost of the extra complexities induced by nonlocal features is rooted on the observation that in many mechanical circumstances the process zone, where decohesion develops, involves a spatially extended microstructure which produces complex bridging spatial effects. Typically, spatial constitutive interaction can be effectively modelled by integral nonlocal models. Along the paper the interface constitutive relations are developed following a thermodynamic consistent approach and the main features of the proposed approach can be summarized as: *(i)* Nonlocal elasticity removes the stress singularity and smooths stress distributions near the crack tip; *(ii)* Nonlocal damage interface ensures regularization and then no solution jumps are produced and a mesh objective solution is expected, even without invoking viscous regularization procedures. The evolution of the damage along the interface, and the subsequent decohesion, are driven by nonlocal damage laws. Namely a spatial average (nonlocal) energy release rate is responsible for the local damage activation function and the damage flow rules are nonlocal as well. The paper deals only with the general theoretical framework of the model leaving out of the presentation, for lack of space, relevant finite element implementation and specific numerical results. Both topics are the subject of an ongoing research activity.

1. INTRODUCTION

In many mechanical situations the presence of continuous joints connecting elements is observed. A meaningful analysis requires an accurate mechanical characterization of the joints, usually modelled as interfaces connecting deformable bodies. Along the interfaces elastic and inelastic phenomena develop such as strains localization or displacement discontinuities, decohesion, sliding, rate dependency, etc. All the above phenomena change the behaviour of the entire system and eventually affect the ultimate structural failure mode.

Examples of the use of interface models in structural problems can be found for masonry structures (Giambanco et al. [1]) or for rock block interactions [Giambanco and Mròz [2]). Interfaces play a central role for laminate composite structures; in fact, such kind of structures are prone to delamination weakness. Many interface formulations deal with composite delamination problems, using fracture (Point and Sacco [3]), or alternatively damage approach (Allix et al. [4], Alfano and Cristfield [5]). Interface Damage Mechanics (Chaboche et al [6]), applied to delamination problems, has been proved to be an efficient and flexible tool capable to describe formation, development and propagation of delamination. However, it has been shown that for avoiding the possibility of sudden *solution jumps* some regularization technique is required, which is usually of viscous nature [6].

One of the main aspects emerging in modern constitutive modelling of materials, particularly when observed at small scales, is the crucial role of the micro-structure, i.e. inhomogeneity of components and presence of defects. Micro-structure might affects both the reversible elastic behaviour and the path of formation and development of localized damage. The actual discrete nature of the material often requires the application of nonlocal constitutive relations. In nonlocal models the stress in a material point is not directly related to the kinematical state (strains and internal variables) at the same point, but rather it depends on the kinematical state in a finite size neighbour region. Recently nonlocal elasticity models of Eringen type (e.g. Eringen [7], Bažant and Jirásek [8]) have been fully framed in a thermodynamic consistent framework (Polizzotto [9]) and also the failure localization of damaging structures have been reformulated following nonlocal thermodynamic arguments (Borino et al [10]). Very recently elasticity and damage nonlocal coupling has been investigated by Polizzotto[11]. Beside an introductory paper (Borino et. al [12]), in the authors' knowledge nonlocal approaches have not been considered for interface modelling so far, probably because structures with interface do not need a regularization for the loose of ellipticity condition, as for continuum softening media. For interfaces, nonlocal models should be adopted with the only intent to better reproduce an actual complex material behaviour and for a more accurate valuation of the stress distribution.

The main subject of this paper is the presentation of the fundamentals of an integral nonlocal formulation for both elasticity and damage in a coupled form.

2. NONLOCAL INTERFACE MODEL

Let us consider an interface layer in which all the mechanical properties are projected over a surface of zero thickness. Figure 1. shows a a simple 2-D structure with a 1-D interface.

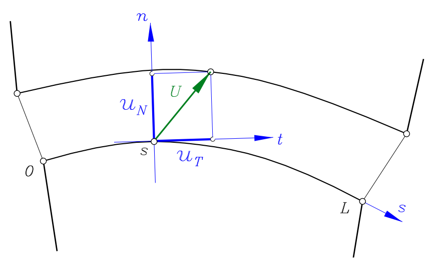

Figure 1. Sketch of a curved interface with it's local relative displacements variables.

The kinematics of the interface is described by the relative displacement $[\boldsymbol{u}] \equiv \boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$, where $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are the displacements of the two opposite sides of the interface. For the case of 1-D interfaces, the relative vector displacement $\boldsymbol{u}$ can be decomposed in two components $u_I(s)$, where s is a curvilinear coordinate, ranging between 0 and L. The index I can assume the values N and T, denoting respectively the relative displacement component along the normal direction $\boldsymbol{n}(s)$ and the one along the tangential direction $\boldsymbol{t}$.

In order to take into account elastic interactions among different points on the interface, the deformation state, beside the local relative displacement $u_I(s)$, is defined by a further nonlocal integral averaging measure of the spatial displacement differences, [11], given as

$$\overline{\Delta u}_I(s) = \mathcal{A}(\Delta u_I) \equiv \frac{1}{\Omega^e_\infty} \int_0^L \alpha_e(r,s)\,[u_I(r) - u_I(s)]\,dr, \tag{1}$$

where $\alpha(r)$ is a positive two-point spatial weight function which is symmetric, i.e. $\alpha_e(r,s) = \alpha_e(s,r) = \alpha_e(|s-r|)$ and $d\alpha_e(|r|)/dr \leq 0$. A possible choice for $\alpha(r,s)$ is the Gauss error function $\alpha_e(r) = C\exp(-r^2/\ell_e^2)$, where C is a normalization factor and ℓ_e is the internal length, a material parameter controlling the length of spatial interaction effects. In eq. (1) the following definition have been adopted

$$\Omega^e(r) = \int_0^L \alpha(|r-s|)\,dr; \qquad \Omega^e_\infty = \int_{-\infty}^{\infty} \alpha(|r|)\,dr, \tag{2a, b}$$

which are measure of the representative volume. A nonlocal model based on the spatial difference was introduced by Borino, [10], for continuum damage problems and by Polizzotto, [9], [11], for elasticity problems. The nonlocal operator $\mathcal{A}$ is self-adjoint and gives no contribution for $u_I(s)$ constant. Figure 2 shows a sketch of the interface nonlocal regularization features with respect to the opening displacement field $u_N(r)$ along a linear interface.

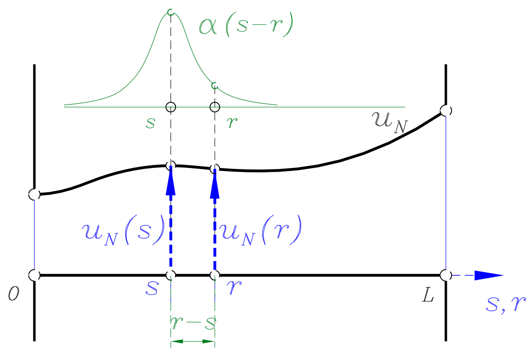

Figure 2. Sketch of the nonlocal difference displacement correction field $\overline{\Delta u}_N$.

The damage along the interface is described by two internal variables, $\omega_I(r)$, ranging between 0 (integrity) to 1 (fully damaged). Nonlocal damage fields are defined for considering spatial cracking interactions

$$\widetilde{\omega}_I(r) = \mathcal{R}_d(\omega_I) \equiv \left[1 - \frac{\Omega^d(r)}{\Omega^d_\infty}\right]\omega_I(r) + \frac{1}{\Omega^d_\infty}\int_0^L \alpha_d(r,s)\,\omega_I(s)\,ds, \qquad I = N, T \tag{3}$$

where similar positions of eqs. (3) are adopted,with the only difference given by the damage internal length parameter $\ell_d \neq \ell_e$, being related to the spatial damage interactions.

3. THERMODYNAMIC FRAMEWORK

In order to achieve a model which a-priori complies with thermodynamic requirements, the following Helmholtz free energy functional density (for unit surface) is assumed:

$$\psi(u_N, u_T, \overline{\Delta u}_N, \overline{\Delta u}_T, \widetilde{\omega}_N, \widetilde{\omega}_T, \xi) = \psi_e(u_N, u_T, \overline{\Delta u}_N, \overline{\Delta u}_T, \widetilde{\omega}_N, \widetilde{\omega}_T) + \psi_{in}(\xi), \tag{4}$$

where ψ_e and ψ_{in} are the elastic-damage and the internal free energy. ξ is an internal variable describing the damage hardening. We assume ψ_e in the following form

$$\begin{aligned}\psi_e = {}& \frac{1}{2}(1-\widetilde{\omega}_N)\, H(u_N)\, K_N^+\, u_N^2 + \frac{1}{2} H(-u_N)\, K_N^-\, u_N^2 + \frac{1}{2}(1-\widetilde{\omega}_T)\, K_T\, u_T^2 \\ & + \frac{1}{2}(1-\widetilde{\omega}_N)\, H(u_N)\, K_N^+\, \overline{\Delta u}_N^2 + \frac{1}{2} H(-u_N)\, K_N^-\, \overline{\Delta u}_N^2 + \frac{1}{2}(1-\widetilde{\omega}_T)\, K_T\, \overline{\Delta u}_T^2,\end{aligned} \tag{5}$$

where $H(x)$ is the Heaveside function, i.e. $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ for $x < 0$. The state equations can be derived from the free energy function of eq. (5)

$$\bar{\sigma}_N := \frac{\partial \psi}{\partial u_N} = \left[(1-\widetilde{\omega}_N)\, H(u_N)\, K_N^+ + H(-u_N)\, K_N^-\right] u_N; \qquad \bar{\sigma}_T := \frac{\partial \psi}{\partial u_T} = (1-\widetilde{\omega}_N) K_T u_T; \tag{6a, b}$$

$$\overline{\Delta\sigma}_N := \frac{\partial \psi}{\partial \overline{\Delta u}_N} = \left[(1-\widetilde{\omega}_N)\, H(u_N)\, K_N^+ + H(-u_N)\, K_N^-\right] \overline{\Delta u}_N \equiv K_N^*\, \overline{\Delta u}_N; \tag{6c}$$

$$\overline{\Delta\sigma}_T := \frac{\partial \psi}{\partial \overline{\Delta u}_T} = (1-\widetilde{\omega}_N) K_T\, \overline{\Delta u}_T \equiv K_T^*\, \overline{\Delta u}_T; \tag{6d}$$

$$\bar{Y}_N := -\frac{\partial \psi}{\partial \widetilde{\omega}_N} = \frac{1}{2}\, H(u_N)\, K_N^+ (u_N^2 + \overline{\Delta u}_N^2); \qquad \bar{Y}_T := -\frac{\partial \psi}{\partial \widetilde{\omega}_T} = \frac{1}{2}\, K_T^+ (u_T^2 + \overline{\Delta u}_T^2) \tag{7a, b}$$

$$\chi := \frac{\partial \psi_{in}}{\partial \xi} \tag{8}$$

The stresses defined by eqs.(6), i.e. the stresses pertaining to the constitutive elastic relations, are not the *true* Cauchy stresses to be inserted in the equilibrium equations (Borino and Polizzotto [13]). The true stresses σ_N, σ_T are derived by the second thermodynamic principle as the Clausius-Duhem inequality enforced globally along the interface.

$$W \equiv \int_0^L D\, ds = \int_0^L \left(\sigma_N \dot{u}_N + \sigma_T \dot{u}_T - \dot{\psi}\right) ds. \tag{9}$$

Developing $\dot{\psi}$ from eq. (5), taking into account eqs. (6) and (7)

$$\begin{aligned}\dot{\psi} &= \frac{\partial \psi}{\partial u_N}\dot{u}_N + \frac{\partial \psi}{\partial u_T}\dot{u}_T + \frac{\partial \psi}{\partial \overline{\Delta u}_N}\overline{\Delta \dot{u}}_N + \frac{\partial \psi}{\partial \overline{\Delta u}_T}\overline{\Delta \dot{u}}_T + \frac{\partial \psi}{\partial \widetilde{\omega}_N}\dot{\widetilde{\omega}}_N + \frac{\partial \psi}{\partial \widetilde{\omega}_T}\dot{\widetilde{\omega}}_T + \frac{\partial \psi}{\partial \xi}\,\dot{\xi} \\ &= \bar{\sigma}_N\, \dot{u}_N + \bar{\sigma}_T\, \dot{u}_T + \overline{\Delta\sigma}_N\, \overline{\Delta \dot{u}}_N + \overline{\Delta\sigma}_T\, \overline{\Delta \dot{u}}_T + \bar{Y}_N\, \dot{\widetilde{\omega}}_N - \bar{Y}_T\, \dot{\widetilde{\omega}}_T + \chi\, \dot{\xi}\end{aligned} \tag{10}$$

which substituted in eq. (9) gives

$$W = \int_0^L (\sigma_N\, \dot{u}_N + \sigma_T\, \dot{u}_T - \bar{\sigma}_N\, \dot{u}_N - \bar{\sigma}_T\, \dot{u}_T - \overline{\Delta\sigma}_N\, \overline{\Delta \dot{u}}_N - \overline{\Delta\sigma}_T\, \overline{\Delta \dot{u}}_T + \bar{Y}_N\, \dot{\widetilde{\omega}}_N + \bar{Y}_T\, \dot{\widetilde{\omega}}_T - \chi\, \dot{\xi}) ds \geq 0. \tag{11}$$

The second principle can be alternatively enforced locally (pointwise) as

$$D=\sigma_N\dot{u}_N+\sigma_T\dot{u}_T-\bar{\sigma}_N\dot{u}_N-\bar{\sigma}_T\dot{u}_T-\overline{\Delta\sigma}_N\overline{\Delta\dot{u}}_N-\overline{\Delta\sigma}_T\overline{\Delta\dot{u}}_T+\bar{Y}_N\dot{\tilde{\omega}}_N+\bar{Y}_T\dot{\tilde{\omega}}_T-\chi\,\dot{\xi}+P_e+P_d\geq 0, \tag{12}$$

where P_e and P_d are nonlocal residual functions related to elasticity and damage processes.

3.1 Nonlocal elastic deformation process

The inequality (12) holds for every deformation process, including the elastic nondissipative ones for which $\dot{\tilde{\omega}}_N=\dot{\tilde{\omega}}_T=\dot{\xi}=P_d=0$ and then eq. (12) particularizes

$$D=\sigma_N\dot{u}_N+\sigma_T\dot{u}_T-\bar{\sigma}_N\dot{u}_N-\bar{\sigma}_T\dot{u}_T-\overline{\Delta\sigma}_N\overline{\Delta\dot{u}}_N-\overline{\Delta\sigma}_T\overline{\Delta\dot{u}}_T+P_e=0, \tag{13}$$

which integrated along the interface, considering the insulation condition $\int_0^L P_e\,ds=0$, gives

$$\int_0^L\left(\sigma_N\dot{u}_N+\sigma_T\dot{u}_T-\bar{\sigma}_N\dot{u}_N-\bar{\sigma}_T\dot{u}_T-\overline{\Delta\sigma}_N\overline{\Delta\dot{u}}_N-\overline{\Delta\sigma}_T\overline{\Delta\dot{u}}_T\right)\,ds=0. \tag{14}$$

Considering the definition (1), the following identity can be developed

$$\begin{aligned}\int_0^L\overline{\Delta\sigma}_I\,\overline{\Delta\dot{u}}_I\,ds&=\int_0^L\overline{\Delta\sigma}_I(s)\,\mathcal{A}(\Delta\dot{u}_I)\,ds\equiv\int_0^L\overline{\Delta\sigma}_I(s)\,\frac{1}{\Omega_\infty^e}\int_0^L\alpha_e(s,r)\,\left[\dot{u}_I(r)-\dot{u}_I(s)\right]\,dr\,ds\\&=\int_0^L\!\left(\frac{1}{\Omega_\infty^e}\int_0^L\alpha_e(s,r)\,\left[\overline{\Delta\sigma}_I(r)-\overline{\Delta\sigma}_I(s)\right]\,dr\right)\dot{u}_I(s)\,ds\equiv\int_0^L\mathcal{A}(\overline{\Delta\sigma}_I)\dot{u}_I(s)\,ds\end{aligned} \tag{15}$$

which substituted into eq. (14) gives

$$\int_0^L\Big\{\,\left[\sigma_N-\bar{\sigma}_N-\mathcal{A}(\overline{\Delta\sigma}_N)\right]\,\dot{u}_N+\left[\sigma_T-\bar{\sigma}_T-\mathcal{A}(\overline{\Delta\sigma}_T)\right]\,\dot{u}_T\Big\}ds=0. \tag{16}$$

Equation (16) is true for every field $\dot{u}_I(s)$, then the interface Cauchy stresses reads

$$\sigma_I=\bar{\sigma}_I+\mathcal{A}(\overline{\Delta\sigma}_I).\qquad I=N,T. \tag{17}$$

Having the interest of expressing the stresses in terms of the displacement fields, it is then necessary to insert in eq. (17) the state laws given by eqs. (7), namely

$$\sigma_I=K_I^*\,u_I+\mathcal{A}\Big(K_I^*(\overline{\Delta u}_I)\Big)=K_I^*\,u_I+\mathcal{A}\Big(K_I^*\mathcal{A}(\Delta u_I)\Big). \tag{18}$$

Expanding the last term

$$\mathcal{A}\Big(K_I^*\mathcal{A}(\Delta u_I)\Big)=\int_0^L J_I(s,r)\,\Big[u_I(r)-u_I(s)\Big]\,dr, \tag{19}$$

where

$$J_I(s,r)=\int_0^L\frac{1}{\Omega_\infty^{e2}}\alpha_e(s,t)\,\alpha_e(t,r)\,K_I^*(t)\,dt-\frac{1}{\Omega_\infty^e}\alpha_e(s,r)\Big[\gamma_e(r)\,K_I^*(r)+\gamma_e(s)\,K_I^*(s)\Big] \tag{20}$$

$$\gamma_e(s)=\frac{1}{\Omega_\infty^e}\int_0^L\alpha_e(s,r)\,dr, \tag{21}$$

then the nonlocal interface “true” stress – strain relation is

$$\sigma_I(s) = K_I^*(s)\, u_I(s) + \int_0^L J_I(s,r)\, \Big[u_I(r) - u_I(s)\Big]\, dr, \tag{22}$$

The relation (22) can be rewritten in alternative form as

$$\sigma_I(s) = \Big[K_I^*(s) - \bar{K}_I^*(s)\Big]\, u_I(s) + \int_0^L J_I(s,r)\, u_I(r)\, dr, \qquad (23).$$

where

$$\bar{K}_I^*(s) = \frac{1}{\Omega_\infty^{e2}} \int_0^L\!\!\int_0^L \alpha_e(s,t)\, \alpha_e(t,r)\, K_I^*(t)\, dt\, dr - \frac{1}{\Omega_\infty^e} \int_0^L \alpha_e(s,r)\, \gamma_e(r)\, dr - \gamma_e^2(s)\, K_I^*(s). \tag{24}$$

It is to remark that a similar interpretation given in [12] can be applied to eq. (23). In fact, for a homogenous elastic material $K_I^*(s) = \text{const.}$, the first local term is a boundary correction term which goes to zero at points s far from the interface boundaries, $s = 0$ and $s = L$. However, in the most common case, because of the inhomogeneity induced by the development of damage (and also because of the different interface stiffness in traction and compression state) the first local term is effective also far from the boundaries and gives an important contribution for the spatial transition states. Examining eq. (22), as pointed out by Polizzotto [11], it emerges that for uniform state of deformation $u_I(s)$ =const., no matter of the etherogeneity induced by the damage distribution, the stress is $\sigma_I(s) = K_I^*(s)\, u_I$ which is the actual relation for a purely local material, i.e. nonlocality is induced solely by nonhomogeneous state of deformation.

The nonlocal residual function related to the elasticity can be recovered from eq. (13)

$$P_e = \overline{\Delta\sigma}_N\, \mathcal{A}(\Delta\dot{u}_N) - \mathcal{A}(\overline{\Delta\sigma}_N)\, \dot{u}_N + \overline{\Delta\sigma}_T\, \mathcal{A}(\Delta\dot{u}_T) - \mathcal{A}(\overline{\Delta\sigma}_T)\, \dot{u}_T \tag{25}$$

3.2 Nonlocal damage deformation process

Let us now consider an elastic-damage process. The dissipation (12), considering that the relations (17) hold also for an elastic damaging process, transforms as

$$D = \bar{Y}_N \dot{\tilde{\omega}}_N + \bar{Y}_T \dot{\tilde{\omega}}_T - \chi\dot{\xi} + P_d \geq 0. \tag{26}$$

Following the same arguments given in [12], we introduce the hypothesis that the dissipation can be alternatively expressed as a bilinear form of the local fluxes $\dot{\omega}_I$, $\dot{\xi}$, namely

$$D = X_N \dot{\omega}_N + X_T \dot{\omega}_T - \chi\dot{\xi} \geq 0. \tag{27}$$

where X_I are the relevant variables (of nonlocal nature) to be thermodynamically associated to the (local) damage fluxes $\dot{\omega}_I$. Comparing eqs. (26) and (27) it follows that

$$P_d = X_N \dot{\omega}_N - \bar{Y}_N \dot{\tilde{\omega}}_N + X_T \dot{\omega}_T - \bar{Y}_T \dot{\tilde{\omega}}_T. \tag{28}$$

Integrating eq. (28) along the all interface length and invoking the damage energy insulation condition the following relation is obtained

$$\int_0^L P_d\, ds = \int_0^L \big(X_N \dot{\omega}_N - \bar{Y}_N \dot{\tilde{\omega}}_N + X_T \dot{\omega}_T - \bar{Y}_T \dot{\tilde{\omega}}_T\big)\, ds = 0, \tag{29}$$

next, substituting the nonlocal definition of $\dot{\tilde{\omega}}_I$, given in eq. (4), we obtain

$$\begin{aligned}\int_0^L \Big\{ X_N(s)\, \dot{\omega}_N(s) - \bar{Y}_N(s) \left[\left(1 - \frac{\Omega^d(s)}{\Omega^d_\infty}\right) \dot{\omega}_N(s) + \frac{1}{\Omega^d_\infty} \int_0^L \alpha_d(s,r)\, \dot{\omega}_N(r)\, dr \right] \\ + X_T(s)\, \dot{\omega}_T(s) - \bar{Y}_T(s) \left[\left(1 - \frac{\Omega^d(s)}{\Omega^d_\infty}\right) \dot{\omega}_T(s) + \frac{1}{\Omega^d_\infty} \int_0^L \alpha_d(s,r)\, \dot{\omega}_T(r)\, dr \right] \Big\} ds = 0\end{aligned} \tag{30}$$

which can be rewritten as

$$\begin{aligned}\int_0^L \Bigg\{ \left[X_N(s) - \left(1 - \frac{\Omega^d(s)}{\Omega^d_\infty}\right) \bar{Y}_N(s) + \frac{1}{\Omega^d_\infty} \int_0^L \alpha_d(s,r)\, \bar{Y}_N(r)\, dr \right] \dot{\omega}_N(s) \\ + \left[X_T(s) - \left(1 - \frac{\Omega^d(s)}{\Omega^d_\infty}\right) \bar{Y}_T(s) + \frac{1}{\Omega^d_\infty} \int_0^L \alpha_d(s,r)\, \bar{Y}_T(r)\, dr \right] \dot{\omega}_T(s) \Bigg\} ds = 0.\end{aligned} \tag{31}$$

Equation (31) must be satisfied for every damage rate field $\dot{\omega}_I$ and then

$$X_I(s) \equiv \widetilde{Y}_I(s) = \left(1 - \frac{\Omega^d(s)}{\Omega^d_\infty}\right) \bar{Y}_I(s) + \frac{1}{\Omega^d_\infty} \int_0^L \alpha_d(s,r)\, \bar{Y}_I(r)\, dr \tag{32}$$

furthermore considering the state equations (8) one obtain

$$\widetilde{Y}_I(s) = \left[1 - \frac{\Omega^d(s)}{\Omega^d_\infty}\right] \frac{1}{2} K_I^* \big(u_I^2 + \overline{\Delta u}_I^2\big) + \frac{1}{2\Omega^d_\infty} \int_0^L \alpha_d(s,r)\, K_I^* \big(u_I^2(r) + \overline{\Delta u}_I^2(r)\big)\, dr, \tag{33}$$

from which the nonlocal elasticity–nonlocal damage coupling effect can be envisaged. The expression of the damage residual function of eq (28) reads

$$P_d = \big(\widetilde{Y}_N \dot{\omega}_N - \bar{Y}_N \dot{\tilde{\omega}}_N\big) + \big(\widetilde{Y}_T \dot{\omega}_T - \bar{Y}_T \dot{\tilde{\omega}}_T\big). \tag{34}$$

The dissipation inequality involving only local flow variables is

$$D = \widetilde{Y}_N\, \dot{\omega}_N + \widetilde{Y}_T\, \dot{\omega}_T - \chi\, \dot{\xi} \geq 0. \tag{35}$$

Since in eq. (35) the variables associated to the local fluxes are $\widetilde{Y}_N$, $\widetilde{Y}_T$ and χ, the damage activation relation must be a function of these parameters, namely

$$\phi_d(\widetilde{Y}_N, \widetilde{Y}_T, \chi) = g(\widetilde{Y}_N, \widetilde{Y}_T) - \chi - Y_0 \leq 0 \qquad \text{for all } 0 \leq s \leq L \tag{36}$$

where $g(\widetilde{Y}_N, \widetilde{Y}_T)$ is an homogeneous function and Y_0 is the initial damage activation threshold. Finally, under the generalized associativity hypothesis, the damage activation function is also a potential function and the flow rules are of a generalized normality type

$$\dot{\omega}_N = \frac{\partial \phi_d}{\partial \widetilde{Y}_N} \dot{\lambda}_d, \qquad \dot{\omega}_T = \frac{\partial \phi_d}{\partial \widetilde{Y}_T} \dot{\lambda}_d, \qquad \dot{\xi} = -\frac{\partial \phi_d}{\partial \chi} \dot{\lambda}_d, \tag{37}$$

where $\dot{\lambda}_d \geq 0$ is the damage activation multiplier. The interface damage constitutive relations are then completed by the usual loading/unloading conditions $\phi_d \, \dot{\lambda}_d = \dot{\phi}_d \, \dot{\lambda}_d = 0$

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