ICF11 Long Abstract

DYNAMIC ESHELBY TENSORS AND ASSOCIATED SOLUTION OF THE INHOMOGENEOUS HELMHOLTZ EQUATION FOR ELLIPSOIDAL INCLUSIONS

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In this paper a compact two-dimensional integral representation for the *dynamic* variant of Eshelby tensor for the three-dimensional infinite space with linear elastic isotropic properties is derived [+]. For this representation it is crucial to have an appropriate approach for the solution of the inhomogeneous Helmholtz equation for an ellipsoidal homogeneous source region which represents the inclusion. The integral formulation for the solution of the inhomogeneous Helmholtz equation holds for the inside and outside region of an ellipsoidal inclusion (source region of unit density). Using this integral formulation closed form expressions are obtained for special cases such as spheres and continuous fibers, coinciding with results obtained by Mikata and Nemat-Nasser (1990)[++] and Michelitsch et. al (2001)[+++] by employing other techniques, respectively. The inclusion is assumed to undergo a spatially uniform time harmonic eigenstrain transformation

$$\boldsymbol{\epsilon}^* = \Theta_s(\mathbf{r})\boldsymbol{\epsilon}^0 e^{i\omega t} \tag{1}$$

where $\Theta_s(\mathbf{r})$ denotes the characteristic function of the inclusion and $\boldsymbol{\epsilon}^0$ a constant symmetric second rank tensor. Then the strain writes[1]

$$\epsilon_{il}(\mathbf{r}, \boldsymbol{\xi}, \omega) = -C_{kjrs}\epsilon^0_{rs}(P_{ijkl}(\mathbf{r}, \boldsymbol{\xi}, \omega))_{(il)} \tag{2}$$

where (li) indicates symmetrization with respect to the subscripts il and

$$P_{ijkl}(\mathbf{r}, \boldsymbol{\xi}, \omega) = \partial_i \partial_j G_{kl}(\mathbf{r}, \boldsymbol{\xi}, \omega) \tag{3}$$

where

$$\boldsymbol{G}(\mathbf{r}, \boldsymbol{\xi}, \omega) = \frac{1}{\rho\omega^2}[\beta_2^2 g(\mathbf{r}, \boldsymbol{\xi}, \beta_2)\mathbf{1} - \nabla \otimes \nabla\{g(\mathbf{r}, \boldsymbol{\xi}, \beta_1) - g(\mathbf{r}, \boldsymbol{\xi}, \beta_2)\}] \tag{4}$$

with the two frequencies β_s given by

$$\beta_1 = \frac{\omega}{c_1} = \omega\sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad \beta_2 = \frac{\omega}{c_2} = \omega\sqrt{\frac{\rho}{\mu}} \tag{5}$$

corresponding to one longitudinal and two transversal acoustic waves in an isotropic 3D medium where λ and μ denote the Lame constants. The functions (Helmholtz-potentials) $g(\mathbf{r}, \boldsymbol{\xi}, \beta_i)$ are the solutions of the scalar inhomogeneous Helmholtz equation

$$(\Delta + \beta^2)g(\mathbf{r}, \boldsymbol{\xi}, \beta) + \Theta_s(\mathbf{r}) = 0 \tag{6}$$

Using (2) we define the *dynamic Eshelby tensor* $\boldsymbol{S}$ analogously to statics by (Mikata and Nemat-Nasser 1990)$^{++}$

$$\epsilon_{il}(\mathbf{r}, \boldsymbol{\xi}, \omega) = S_{ilrs}(\mathbf{r}, \boldsymbol{\xi}, \omega)\epsilon^0_{rs} \tag{7}$$

where the *dynamic Eshelby tensor* is given by

$$S_{ilrs}(\mathbf{r}, \boldsymbol{\xi}, \omega) = -C_{kjrs}(P_{ijkl}(\mathbf{r}, \boldsymbol{\xi}, \omega))_{(il)} \tag{8}$$

This formulation holds for inclusions of arbitrary shape. In this paper we will consider an ellipsoidal source region.

Unlike in statics, in the dynamic case P_{ijkl} is a spatially *non-uniform* tensor function inside an ellipsoidal inclusion. When we expand P_{ijkl} in a series with respect to ω, the zero order in ω corresponds to the static limit leading

[1]where the phase factor $e^{i\omega t}$ is skipped in the following

to the classical result of Eshelby (1957). For the inside region of inclusion the static limiting case is explicitly confirmed [+].

The derived formulation for the *dynamic* Eshelby tensor may be useful for application in a wide range of dynamical problems, such as the dynamical behavior of cracks and the effective dynamic characteristics in composite materials.

[+]Michelitsch T.M., Gao H., Levin V.M., 2003. Dynamic Eshelby Tensors and Potentials for Ellipsoidal Inclusions, Proceedings of the Royal Society of London A, 459, 863-890.

[++] Mikata Y., Nemat-Nasser S. 1990. Elastic field due to a Dynamically Transforming Spherical Inclusion. J. Appl. Mech. ASME 57 (4): 845-849.

[+++] Michelitsch T.M., V.M. Levin and H. Gao, 2002. Dynamic Green's Functions of a Quasiplane Piezoelectric Medium With Inclusion. Proceedings of the Royal Society of London A, 458, 2393-2415.