NEAR-TIP ASYMPTOTIC ANALYSIS OF A PKN FLUID-DRIVEN FRACTURE WITH NON-LOCAL ELASTICITY EQUATION

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ABSTRACT

One of the most widely used hydraulic fracturing models in the oil and gas industry is the so-called PKN model (named after Perkins, Kern and Nordgren). Based on certain geometrical assumptions, in the PKN model the elasticity equation (which relates the fracture width with the internal fluid pressure) is reduced to a local operator. This local operator imposes a zero pressure boundary condition at the fracture tips, which is incompatible with the classical square-root tip asymptote. This limitation reduces the applicability of the PKN model to situations in which the rock toughness is negligible. In this paper, we propose an alternative, non-local formulation for the elasticity equation of a PKN-type fracture. We also perform a near-tip asymptotic analysis of the new equation, and show the presence of a "plane-strain" zone near the advancing tip, in which the appeareance of fluid pressure singularities (similar to those obtained in pure plane-strain fracture models) is predicted.

1 INTRODUCTION

Hydraulic fracturing is one of the main techniques employed by the oil and gas industry to increase the productivity of hydrocarbon reservoirs. Modeling of hydraulic fractures (specifically for oil and gas applications) dates back to the 1950s. Due to the complexity of the coupled mechanisms that control the propagation of hydraulic fractures (the deformation induced by the fluid pressure on the rock, the flow of fluid within the fracture, and the fracture propagation criterion) the use of idealized models became necessary for studying this process. One of the first (and still more widely used) of such idealized models is the so-called "PKN fracture" (named after Perkins & Kern [1], Nordgren [2], later modified by Kemp [3]).

2 THE PKN HYDRAULIC FRACTURE MODEL

A schematic of the geometry of the PKN model is depicted in Fig. 1: a hydraulic fracture propagates along the x-axis while fully contained within a rock layer of constant thickness $2H$, with elastic properties defined by its Young's modulus E and Poisson's ratio ν. Propagation is assumed to be symmetric with respect to the wellbore (represented by the y-axis), and perpendicular to the minimum *in situ* stress σ_o. It is assumed that (i) the fracture half-length ℓ is much greater than its height; and (ii) the fracture width varies slowly along the propagation axis of the fracture. From these, it is concluded that an approximate state of plane strain prevails in planes that are perpendicular to the propagation axis. Thus the fracture shape at any such cross section can be approximated to an elliptical plane-strain fracture with constant internal pressure. We define $w_o(x,t)$ as the maximum width of the fracture at any point of coordinate x, and $p(x,t) = p_{\mathrm{f}}(x,t) - \sigma_o$ as the net fluid pressure, with p_{f} being the absolute fluid pressure.

There are three fundamental equations to be considered in the PKN model: an equation that relates the fluid pressure with the elastic deformation of the solid; an equation that describes the flow of a viscous fluid inside the fracture; and a continuity or fluid volume balance equation. These equations constitute a coupled system that must be solved with a suitable fracture propagation condition.

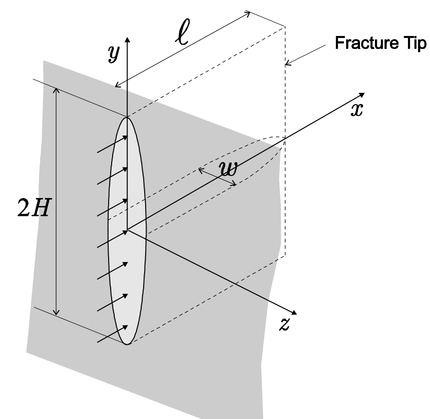

Figure 1: Schematic of the PKN hydraulic fracture model.

For the purpose of this analysis, we are considering only the elasticity equation. Given the particular geometry of the PKN model, Perkins and Kern [1] postulated that the elasticity equation can be expressed in terms of a local operator of the form

$$w_o = M_c \, p \tag{1}$$

where $M_c = 4H/E'$ is the fracture compliance, with $E' = E/(1-\nu^2)$ being the plane-strain Young's modulus. The use of this local elasticity equation greatly reduces the difficulty of solving the PKN model. However, as proved by Kemp [3], this same feature implies that the shape of the advancing tip of a PKN fracture is asymptotically given by $w_o \sim (1 \mp x/\ell)^{1/3}$, which is incompatible with the classical square-root asymptote. This limitation constrains the application of the model to situations in which the toughness of the rock is negligible, compared to other energy dissipation mechanisms.

3 NEW FORMULATION OF THE ELASTICITY EQUATION

An alternative non-local formulation of the elasticity equation for a PKN-type fracture can be derived from the general elasticity equation of a planar fracture (e.g., Hills *et al.* [4]). By retaining the assumptions that the shape of any vertical cross section is elliptical, and that the fluid pressure does not depend on the y coordinate, we obtain

$$\Pi(\xi) = \frac{1}{\pi} \,=\!\!\!\!\!\!\int_{-1}^{1} \frac{\Omega(\xi_o)}{|\xi_o - \xi|} \left[\mathrm{K}\left(-\frac{\beta^2}{(\xi_o-\xi)^2} \right) - \mathrm{E}\left(-\frac{\beta^2}{(\xi_o-\xi)^2} \right) \right] \mathrm{d}\xi_o \tag{2}$$

where $\mathrm{K}(\cdot)$ and $\mathrm{E}(\cdot)$ represent the complete elliptic integrals of the first and second kind, respectively (Abramowitz & Stegun [5]), and $=\!\!\!\!\!\!\int$ indicates the finite part of the hypersingular integral, in the Hadamard sense. In the above, we have introduced the following scaling

$$\beta = \frac{H}{\ell}, \quad \xi = \frac{x}{\ell}, \quad \eta = \frac{y}{\ell}, \quad \Omega = \frac{w_o}{w_*}, \quad \Pi = p\frac{4H}{E' w_*} \tag{3}$$

with w_* being the proper lengthscale for the fracture width (which we leave undefined in this paper). Using integration by parts, (2) can be further reduced to

$$\Pi(\xi) = -\frac{1}{\pi} \fint_{-1}^{1} \operatorname{sgn}(\xi_o - \xi)\ \Omega'(\xi_o)\, \mathrm{E}\left(-\frac{\beta^2}{(\xi_o - \xi)^2}\right) \mathrm{d}\xi_o \tag{4}$$

where sgn(·) represents the sign of the argument, $()'$ indicates the derivative with respect to the argument, and $\fint$ indicates a Cauchy principal integral.

4 ASYMPTOTIC ANALYSIS

4.1 Asymptotic Expansions of the New Elasticity Kernel

We perform an asymptotic analysis of the elasticity equation (4) assuming the dimensionless ratio $\beta = H/\ell \ll 1$, which is a characteristic of a PKN-type fracture. We make use of the following asymptotic expansions for $\mathrm{E}(\cdot)$:

$$\mathrm{E}\left(-\tfrac{1}{\omega^2}\right) \sim \tfrac{1}{\omega} - \tfrac{1}{2}\omega \ln\omega + \left(\tfrac{1}{4} + \ln 2\right)\omega + \tfrac{1}{16}\omega^3 \ln\omega + O\left(\omega^3\right), \ \omega \ll 1, \tag{5a}$$

$$\mathrm{E}\left(-\tfrac{1}{\omega^2}\right) \sim \tfrac{\pi}{2}\left[1 + \tfrac{1}{4}\omega^{-2} - \tfrac{3}{64}\omega^{-4} + \tfrac{5}{256}\omega^{-6} - \tfrac{175}{16384}\omega^{-8}\right] + O(\omega^{-10}), \ \omega \gg 1 \tag{5b}$$

where we can take $\omega = |\rho|/\beta$. These expansions, when truncated after the third term, are actually valid up to $\omega \sim O(1)$. Fig. 2 shows a comparison between the exact kernel and the two expansions.

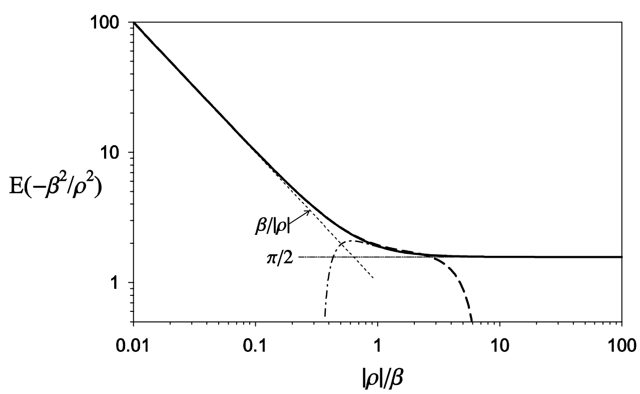

Figure 2: Plot of $\mathrm{E}(-\beta^2/\rho^2)$ (solid line) versus $|\rho|/\beta$. The expansions (5a) and (5b), both truncated at the third term, are plotted for comparison. The first terms of both expansions are also plotted.

Defining $\rho = \xi_o - \xi$, from Fig. 2 it is clear that within a region $\xi_o \in [\xi - \beta/10, \xi + \beta/10]$, the behavior of the elastic kernel is dominated by the term $\beta/|\rho|$, which is the classical plane-strain singularity of the elasticity kernel. For $\rho \gtrsim 10\beta$, we have that $\mathrm{E}\left(-\beta^2/\rho^2\right)$ starts to be dominated by a constant term, i.e., we start to recover the "classical" PKN behavior (local dependency between Π and Ω). We assume that Ω is an analytic, symmetric (even), positive function in $\xi \in (-1, 1)$, with $\Omega(\xi = \pm 1) = 0$. We also assume that the fracture tips are blunt, i.e., that Ω approaches asymptotically the tips as $\Omega \sim A(1 \mp \xi)^\alpha$, $\xi \to \pm 1$, with $0 < \alpha < 1$, and A being an arbitrary positive constant. The gradient Ω' has singular asymptotics of the form $\Omega \sim A\alpha(1 \mp \xi)^{\alpha-1}$, $\xi \to \pm 1$.

4.2 Outer Expansion

Let us first consider the case of a point ξ "away from the tips," which we define as $|\xi| < 1-\beta$. Under the condition of $\beta \ll 1$ and assuming that the gradient Ω' can be approximated via a Taylor's series expansion, we obtain an expansion of the form

$$\begin{aligned}\Pi(\xi;\beta) \quad &\sim \quad \Omega(\xi) + \tfrac{1}{4}\beta^2 \ln\beta\, \Omega''(\xi) + \beta^2 \left[\tfrac{67}{128} - \tfrac{1}{60\pi}\left(\tfrac{16499}{120} + 37\ln 2\right)\right]\Omega''(\xi) + \\ &+ \quad \tfrac{1}{8}\beta^2 \alpha A \left[f_{11}(\xi,\alpha) + f_{12}(\xi,\alpha)\right] + O(\beta^4), \quad |\xi| < 1-\beta. \end{aligned} \tag{6}$$

where the functions f_{11} and f_{12} are defined as

$$f_{11}(\xi,\alpha) \quad = \quad (1+\xi)^{\alpha-2}\left[(\alpha-1)(\gamma+\psi(\alpha)) - \alpha\right], \tag{7a}$$

$$f_{12}(\xi,\alpha) \quad = \quad (1-\xi)^{\alpha-2}\left[(\alpha-1)(\gamma+\psi(\alpha)) - \alpha\right] \tag{7b}$$

with $\psi(\cdot)$ being the digamma function and $\gamma \simeq 0.577216$ is Euler's constant. Notice that as $\beta \to 0$, we recover the "classical" local elasticity equation of Perkins and Kern (in our scaling, $\Pi = \Omega$). A plot of (6) is shown in Fig. 3. To generate this plot, we have used a "test function" $\Omega = (1-\xi^2)^{2/3}$ (i.e., $\alpha = 2/3$ and $A = 2^{2/3}$). For comparison, we have also plotted (dots) the result of a numerical evaluation of (4), which was computed using the software Mathematica™ (© 1988-2000 Wolfram Research, Inc.). The calculations were performed using $\beta = 0.01$. In this plot, the outer expansion has a range of validity of 10β to 100β from the tip (in the example, 100β corresponds to the fracture inlet, i.e., the center of the fracture).

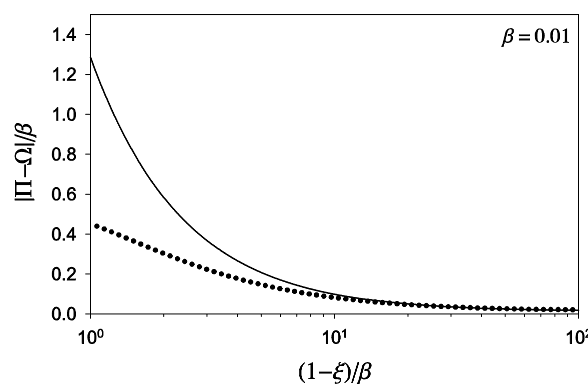

Figure 3: Plot of normalized correction $|\Pi - \Omega|/\beta$ versus normalized distance from the tip $(1-\xi)/\beta$. Comparison of numerical results (dots) versus outer expansion (solid line). Results obtained using $\Omega = (1-\xi^2)^{2/3}$ and $\beta = 0.01$.

4.3 Inner Expansion

If we consider now a point located near one of the fracture tips, i.e., $|\xi| > 1-\beta$, we obtain a different expansion. At this scale the "distance from the tip" $1-\xi$ (if we consider only the right tip $\xi = 1$ for this analysis) becomes smaller than β, and hence the expansion must be performed using $1-\xi$ as a small parameter. If we define

$$\Pi(\xi;\beta) = I_1(\xi;\beta) + I_2(\xi;\beta) \tag{8}$$

expansions for the integrals I_1 and I_2 can be obtained, given by

$$\begin{aligned}\frac{\pi}{A\,\alpha} I_1(\xi;\beta) &\sim \beta\,\pi\cot\pi\alpha\,(1-\xi)^{\alpha-1} + \beta^{\alpha}\lambda_0(\alpha) + \beta^{\alpha-1}\lambda_1(\alpha)\,(1-\xi) + \\ &+ \beta^{-1}\lambda_2(\alpha)\,(1-\xi)^{\alpha+1} + O\left[\beta^{\alpha-2}(1-\xi)^2\right], \ \tfrac{1-\xi}{\beta} \to 0, \qquad (9a)\\ I_2(\xi;\beta) &\sim \tfrac{1}{2}A\left[\beta^{\alpha} + \alpha\,\beta^{\alpha-1}(1-\xi)\right] + \tfrac{1}{8}\alpha A\left\{\beta^2 f_{11} - \beta^{\alpha}\left[\tfrac{1}{\alpha-2} + \tfrac{3}{16(\alpha-4)}\right]\right.\\ &- \left.\beta^{\alpha-1}(1-\xi)\left[\tfrac{\alpha-1}{\alpha-3} - \tfrac{3}{16}\left(\tfrac{\alpha-9}{\alpha-5}\right)\right]\right\} + O\left[\beta^{\alpha-2}(1-\xi)^2\right], \ \tfrac{1-\xi}{\beta} \to 0. \qquad (9b)\end{aligned}$$

Notice that the first term of (9a) represents the known plane-strain pressure singularity for $\Omega \sim A(1-\xi)^{\alpha}$, multiplied by the parameter β. This is the only term of the expansion that is singular in $1-\xi$, and thus it is expected that this singularity should dominate the behavior of Π as $\xi \to 1$. A plot of (8) is shown in Fig. 4. Again, this plot was obtained by using a "test function" $\Omega = (1-\xi^2)^{2/3}$ and $\beta = 0.01$. The plot shows the normalized correction $|\Pi - \Omega|/\beta$ versus the normalized distance from the right tip $(1-\xi)/\beta$. A numerical evaluation of (4) using the software Mathematica$^{\text{TM}}$ has also been included for comparison. Notice that the inner expansion is valid starting at approximately $10^{-2}\beta$. We have also plotted the first term of the expansion $\beta\,\pi\cot\pi\alpha\,(1-\xi)^{\alpha-1}$ (dashed line), which corresponds to the "plane-strain" pressure singularity (Desroches *et al.* [6]). It is evident that the solution converges towards this term for $(1-\xi) \lesssim 10^{-4}\beta$.

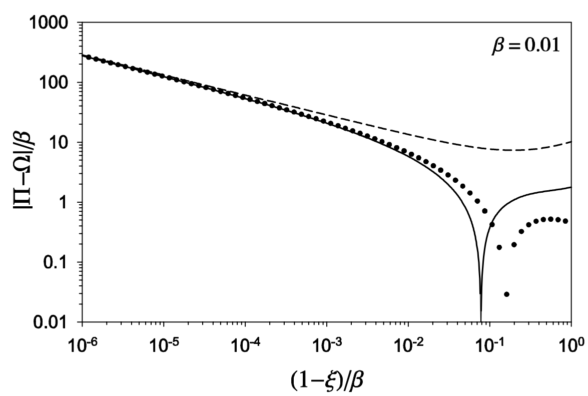

Figure 4: Plot of normalized correction $|\Pi - \Omega|/\beta$ versus normalized distance from the tip $(1-\xi)/\beta$. Comparison of numerical results (dots) versus inner expansion (solid line). The dashed line corresponds to the first term of the expansion. Results obtained using $\Omega = (1-\xi^2)^{2/3}$ and $\beta = 0.01$.

5 CONCLUSIONS

1. A consistent formulation of a non-local elasticity equation for a PKN-type fracture has been introduced. This equation reduces to the "classical" local elasticity equation of the

PKN model when the aspect ratio of the fracture $\beta = H/\ell$ becomes vanishingly small. We have also obtained the correction terms up to order $O(\beta^2)$.

2. The near-tip asymptotic analysis of the proposed elasticity equation reveals the presence of a relatively small region at the fracture tip, in which fracture opening and pressure are related as in a plane-strain fracture. This means that, if we assume a fracture tip shape of the form $(1 \mp \xi)^{\alpha}$, pressure singularities of the form $(1 \mp \xi)^{\alpha-1}$ are observed near the tip.

3. The near-tip pressure singularity indicates that coupling of the new elasticity equation with the rest of the equations of the PKN model (the lubrication and volume balance equations) should yield tip asymptotics similar to those obtained by others in the plane-strain case (Desroches *et al.* [6]; Lenoach [7]; Detournay *et al.* [8]). Hence, this formulation introduces the possibility of adding other parameters to the PKN model, such as rock toughness, leak-off, fluid lag, and distance to a free surface, in a rigorous manner.

4. As the outer solution should correspond to the "classical" PKN solution (which predicts a regular pressure at the tip), and knowing that the resultant inner behavior of the pressure at the tip is singular, a boundary layer (whose thickness has yet to be determined) should form at the tip region. The thickness of this boundary layer should determine the relevance of the pressure singularity in fracture propagation.

5. Some of the approximations of the "classical" PKN model that have been kept for this analysis (such as the constant pressure, elliptical shape assumption, and the approximation of the fracture tip as a "sharp front" without any roundness) should still be validated (using numerical models) and compared against the predictions of the proposed model.

Acknowledgements. The authors would like to thank Schlumberger for permission to publish.

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