ON THE PRESENCE OF *T*-STRESS IN MODE II CRACK PROBLEMS

M. R. Ayatollahi, Mahnaz Zakeri, M.M. Hassani
Department of Mechanical Engineering, Iran University of Science and Technology, Tehran, Iran

ABSTRACT

The elastic stresses in a homogenous cracked body subjected to mixed modes I and II deformation have been derived by Williams as a set of infinite series expansions. The series solution is traditionally divided into symmetric and anti-symmetric components, defined as mode I and mode II, respectively. The first non-singular term in the series solution, often called the "T-stress", is constant and independent of the distance from the crack tip. According to the Williams' definition, the T-stress exists only in mode I and vanishes in mode II. However, recent research studies have revealed several practical applications where significant values of T-stress are present in mode II. In this research, a modified definition for mode I and mode II is suggested in which the presence of the constant stress term in mode II is adopted.

1 INTRODUCTION

There are several methods for deriving the elastic stresses in a cracked body. In one of these methods, the stress components are displayed as a set of series expansions[1]. Williams [1] defined mode I and mode II by separating the series expansions to symmetric and anti-symmetric terms. The elastic stresses then could be derived for each mode from the corresponding part of the solution. The constant term in the series expansions, known as the T-stress, is independent of the distance from the crack tip. This term acts over a large distance from the crack tip; and more knowledge of it, is important for investigating brittle fracture in engineering materials [e.g. 2,3]. For example, it has been observed that when the T-stress in front of a mode I crack is negative, the crack tends to grow along its plane and when the T-stress is positive, the crack deviates from its original plane of growth [2,4].

Based on the Williams' definition [1] for mode II, the T-stress always vanishes in the antisymmetric part of the series solution. This definition has been accepted and used by many researchers [5-8]. However, the results of some recent experimental and analytical investigations [8-10] have revealed many practical applications in which large values of T-stress exist in mode II. Research studies show that ignoring the effect of T-stress in mode II can introduce significant inaccuracies in predicting mode II brittle fracture [10]. This implies that a more comprehensive definition for mode II deformation is required in which the presence of T-stress in mode II can be justifies. In the following sections, the series solution for the elastic stresses in a body containing a sharp crack is elaborated. Then new definitions for mode I and mode II are suggested which are consistent with previous findings and observations.

2 WILLIAMS' EIGEN SERIES SOLUTION

Consider a homogeneous isotropic plate in a state of plane strain or idealized plane stress, which is bounded by two concurrent straight edges defining the solid within the angle 2α (Figure 1). In the absence of body force, the stress components in the polar co-ordinate system are:

$$\sigma_{\theta\theta} = \frac{\partial^2 \Psi}{\partial r^2}, \quad \sigma_{rr} = \nabla^2 \Psi - \sigma_{\theta\theta} \quad , \quad \sigma_{r\theta} = \frac{-\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) \tag{1}$$

where Ψ is the Airy stress function. According to the classical theories of elasticity, Ψ can be found by solving the well known bi-harmonic equation

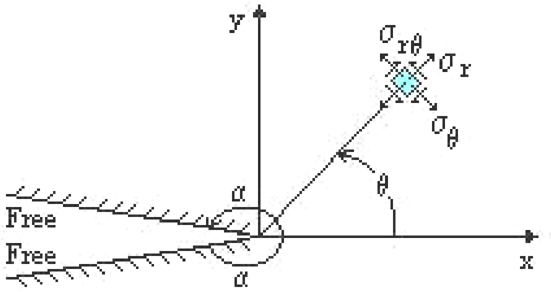

Figure 1: Stresses around the crack tip.

$$\nabla^2(\nabla^2\Psi(r,\theta)) = 0 \qquad , \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2}$$

To simulate a crack, α is assumed to be equal to π. The boundary conditions on the traction free crack surfaces specify that:

$$\sigma_{\theta\theta}(r,\pi) = \sigma_{\theta\theta}(r,-\pi) = 0 \quad , \quad \sigma_{r\theta}(r,\pi) = \sigma_{r\theta}(r,-\pi) = 0 \tag{3}$$

Williams showed that the stress function Ψ can be found as

$$\begin{aligned}\Psi(r,\theta) = &\sum_{n=1,3,\ldots}^{\infty} r^{(\frac{n}{2}+1)}[c_{1n}\{\cos(\frac{n-2}{2})\theta - \frac{n-2}{n+2}\cos(\frac{n+2}{2})\theta\} + c_{2n}\{\sin(\frac{n-2}{2})\theta - \sin(\frac{n+2}{2})\theta\}] \\ &+ \sum_{n=2,4,\ldots}^{\infty} r^{(\frac{n}{2}+1)}[c_{1n}\{\cos(\frac{n-2}{2})\theta - \cos(\frac{n+2}{2})\theta\} + c_{2n}\{\sin(\frac{n-2}{2})\theta - \frac{n-2}{n+2}\sin(\frac{n+2}{2})\theta\}]\end{aligned} \tag{4}$$

The stress function $\Psi(r,\theta)$ can be divided into the even and odd parts with respect to θ. Then the first expression of each summation, the terms containing c_{1n}, represents the symmetric part and the second expression, the terms containing c_{2n}, represents the anti-symmetric part of the solution.

The summations in eqn (4) can be expanded for non-positive values of n. Let's expand them only for n=1, 2. The resulting series expansion is:

$$\Psi(r,\theta) = r^{3/2}[c_{11}\{\cos\left(\frac{\theta}{2}\right) + \frac{1}{3}\cos\left(\frac{3\theta}{2}\right)\} - c_{21}\{\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\}] + r^2[c_{12}\{1-\cos(2\theta)\}] + O(r^{5/2}) \tag{5}$$

where $O(r^{5/2})$ represents the higher order terms which often be ignored near the crack tip. Introducing the function $\Psi(r,\theta)$ to eqn (1), the polar stress components can be obtained:

$$\sigma_{rr} = r^{-1/2}[c_{11}\{\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\} - c_{21}\{\frac{5}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\}] + 2c_{12}\{1-\cos(2\theta)\} + O(r^{1/2}) \tag{6.a}$$

$$\sigma_{\theta\theta} = \frac{3}{4}r^{-1/2}[c_{11}\{\cos\left(\frac{\theta}{2}\right) + \frac{1}{3}\cos\left(\frac{3\theta}{2}\right)\} - c_{21}\{\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\}] + 2c_{12}\{1-\cos(2\theta)\} + O(r^{1/2}) \tag{6.b}$$

$$\sigma_{r\theta} = \frac{1}{4}r^{-1/2}[c_{11}\{\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\} + c_{21}\{\cos\left(\frac{\theta}{2}\right) + 3\cos\left(\frac{3\theta}{2}\right)\}] - 2c_{12}\{\sin(2\theta)\} + O(r^{1/2}) \tag{6.c}$$

which lead to the conventional expressions for elastic stresses near the crack tip:

$$\sigma_{rr} = \frac{1}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)[K_I\{1+\sin^2\left(\frac{\theta}{2}\right)\} + K_{II}\{\frac{3}{2}\sin(\theta) - 2\tan\left(\frac{\theta}{2}\right)\}] + T\cos^2(\theta) + O(r^{1/2}) \tag{7.a}$$

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)[K_I\cos^2\left(\frac{\theta}{2}\right) - \frac{3}{2}K_{II}\sin(\theta)] + T\sin^2(\theta) + O(r^{1/2}) \tag{7.b}$$

$$\sigma_{r\theta} = \frac{1}{2\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)[K_I\sin(\theta) + K_{II}\{3\cos(\theta) - 1\}] - T\sin(\theta)\cos(\theta) + O(r^{1/2}) \tag{7.c}$$

where :

$$K_I = \sqrt{2\pi}\, C_{11} \quad , \quad K_{II} = \sqrt{2\pi}\, C_{21} \quad , \quad T = 4C_{12} \tag{8}$$

K_I and K_{II} which are called the mode I and mode II stress intensity factors, correspond to the well-known singular stresses near the crack tip and the T-stress is independent of distance r from the crack tip.

3 MODE I AND MODE II - CLASSICAL DEFINITIONS

In the previous section, stress components near the crack tip were expressed by a set of series expansions. Williams described these stresses as symmetric and anti-symmetric stress fields representing mode I and mode II, respectively. Based on this definition and using the transformation equations, the elastic stresses derived earlier and presented by eqns. (7a-c) can be rewritten in the Cartesian co-ordinate system as:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)] + T + O(r^{1/2}) \tag{9.a}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)] + O(r^{1/2}) \tag{9.b}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) + O(r^{1/2}) \tag{9.c}$$

for pure mode I or the symmetric part of the solution, and as:

$$\sigma_{xx} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{-\theta}{2}\right)[2 + \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)] + O(r^{1/2}) \tag{10.a}$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right) + O(r^{1/2}) \tag{10.b}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)] + O(r^{1/2}) \tag{10.c}$$

for pure mode II or the anti-symmetric part of the solution.

According to the above equations, the term T is a constant stress parallel to the crack, which exists only in mode I and always vanishes for pure mode II. Although this definition has been accepted and used for many years, it is valid only for limited mode II specimens. In the next section, an improved definition for mode I and mode II is suggested which provides a more accurate expression for the state of stress in mode II.

4 MODE I AND MODE II - MODIFIED DEFINITIONS

The stresses in an elastic cracked body, which was written in the form of the series expansions (eqns. 6.a-c) can be divided into three parts with respect to the distance from the crack tip r: the singular term which is a function of $1/\sqrt{r}$, the constant stress term (independent of r), and the terms containing the higher orders of r (depending on $r^{1/2}$ or higher). The singular term itself contains a symmetric and an antisymmetric component relative to θ.

The in-plane modes of crack deformation can be determined based only on the singular term. The improved definition suggests that pure mode I takes place when the antisymmetric component only in the singular term is zero. In contrast, pure mode II occurs when the symmetric component only in the singular term is zero. Any other combinations of the symmetric and antisymmetric components in the singular term can be attributed to mixed mode I/II. According to this definition, the non-singular terms of stress, including the constant term T and higher order terms $O(r^{1/2})$ are retained in both mode I and mode II. The improved definition is consistent with the practical methods commonly used for determining the modes of crack deformation in engineering problems. For example, in finite element based codes such as ABAQUS, the singular stress components or their corresponding displacement components near the crack tip are used to calculate the ratio of stress intensity factors K_I and K_{II}. Then pure mode I and pure mode II are attributed to the cases where $K_I/K_{II} \to \infty$ and $K_I/K_{II}= 0$, respectively.

As mentioned earlier, in the new definition only the singular terms of the stress components are different for mode I and mode II series solutions, and the other terms are generally the same. Thus the T-stress can be present not only in pure mode I but also in pure mode II. It should be noted that the classical definition for mode II is a very particular case of the new definition in which the T-stress and the symmetric components in the higher order terms are always considered to be zero. This implies that the classical definition is not a comprehensive one.

In the following section, some examples are described for crack specimens having non-zero values of T-stress in pure mode II. These examples can be used to support the new definitions suggested here for mode I and mode II.

5 PRACTICAL SPECIMENS WITH T-STRESS IN MODE II

Among the existing analytical solutions, the T-stress solution for a circular specimen containing an internal crack is studied here. An analytical solution based on the boundary collocation method (BCM) and the Green’s function, is available for the T-stress [8] when the specimen is subjected to diametrically applied concentrated forces. Mixed mode loading of this specimen is shown in Figure 2.

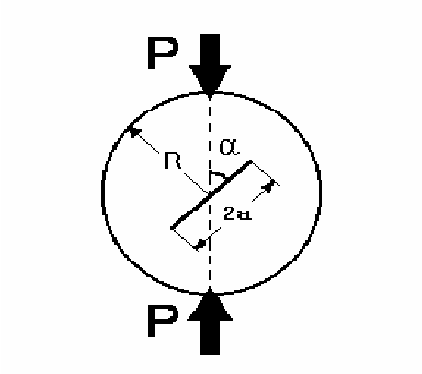

Figure 2: A circular disk containing an angled center crack.

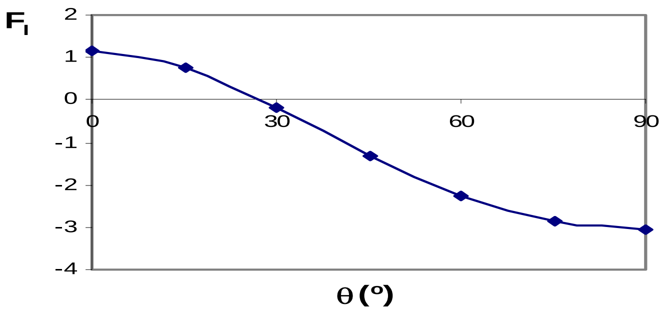

Figure 3: Mode I geometry factor F_I versus crack angle for a/R=0.3 [8].

The stress intensity factors K_I and K_{II} and the related geometry functions F_I and F_{II} were computed with the weight function technique [8]. Figures 3, 4, and 5 show the numerical values of K_I, K_{II} and T in dimensionless forms as F_I, F_{II} and T^*. These values were calculated for the crack length ratio a/R=0.3. Figures 3 and 4 show that when $\theta = 26.9°$, F_I (and consequently K_I) is zero, but F_{II} is non-zero. It means that the specimen is subjected to pure mode II for such a crack angle. Using the numerical results given in Figure 5, it is observed that the T-stress for pure mode II is not zero and has a significant value. This can be clearly justified by the improved definition for mode II but not with the classical definition given by Williams[1].

There are also experimental and numerical results for other crack specimens, which indicate the presence of T-stress in mode II. For example, Ayatollahi et.al. [9,10] determined T-stress for two different mode II test configurations using finite element analysis. Their numerical results again confirm that the T-stress can be present in practical mode II specimens.

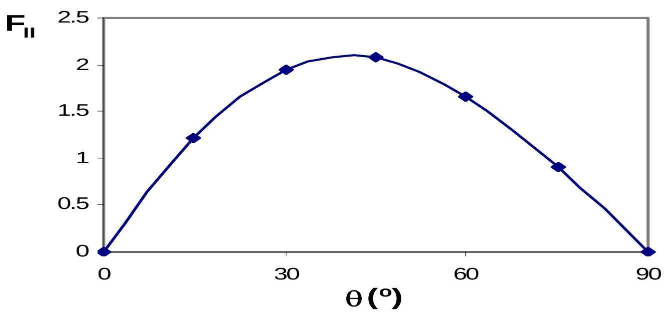

Figure 4: Mode II geometry factor F_{II} versus crack angle for a/R=0.3 [8].

Mixed mode brittle fracture has been studied by many researchers for different materials. Having taken the classical difinitions for deformation modes of a crack, they have often attempted to design test specimens capable of producing symmetric loading for mode I and antisymmetric loading for mode II (*e.g.* [11] and [12]). However, as described in this paper antisymmetric loading is a specific type of shear loading where in addition to K_I, the *T*-stress also vanish [9]. Furthermore, ideal antisymmetric loading rarely occurs for real engineering components and in practice a considerable value of *T*-stress can be present for mode II loading. This suggests that most of the experimental results presented in the literature for mode II brittle fracture can be used only for a limited set of real applications where the crack tip is subjected to conditions very close to antisymmetric loading.

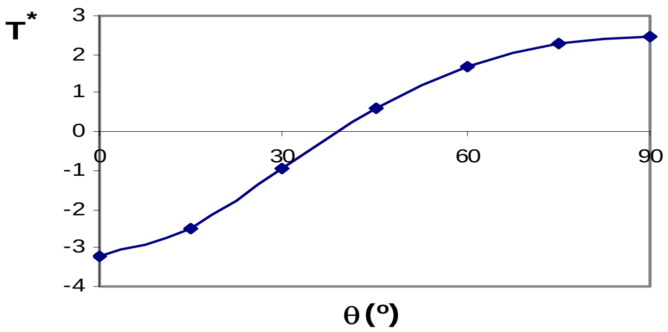

Figure 5 – The dimensionless parameter T^* versus crack angle for a/R=0.3 [8].

6 CONCLUSIONS

In the Williams' solution, the elastic stress field near the crack tip is presented in the form of series expansions and mode I and mode II are defined as the symmetric and anti-symmetric parts of the expansion, respectively. Thus the T-stress exists only in mode I and vanishes in mode II. A new definition for mode I and mode II was suggested which is more accurate than the previous definition. Based on the new definition, the T-stress is not restricted to mode I and can be present in pure mode II.

7 REFRENCES

1- Williams M.L., "On the Stress Distribution at the Base of a Stationary", Journal of Applied Mechanics, pp. 109-114, 1957.
2- Cotterell, B. and J.R. Rice, "Slightly Curved or Kinked Cracks", International Journal of Fracture, Vol. 16, pp. 155-169, 1980.
3- Selvarathinam A.S. and Goree J.G., "T-Stress Based Fracture Model for Cracks in Isotropic Materials", Engineering Fracture Mechanics, Vol. 60, No. 5-6, pp. 543-561, 1998.
4- Richardson D.E. and Goree J.G., "Experimental Verification of a New Two Parameter Fracture Model", Fracture Mechanics: 23th Symposium, ASTM STP 1189, pp. 738-750, 1993.
5- Owen D.R.J. and Fawkes A.J., "Engineering Fracture Mechanics: Numerical Methods and Applications", Pineridge Press Ltd., UK, 1983.
6- Hellan K., "Introduction to Fracture Mechanics", McGraw-Hill, 1984.
7- Fenner R.T., "Engineering Elasticity, application of numerical and analytical techniques", Ellis Horwood, England, 1986.
8- Fett T., "Stress Intensity Factors and T-Stress for Internally Cracked Circular Disks under Various Boundary Conditions", Engineering Fracture Mechanics, Vol. 68, pp. 1119-1136, 2001.
9- Ayatollahi M.R., Pavier M.J. and Smith D.J., "Determination of T-Stress from Finite Element Analysis for Mode I and Mixed Mode I/II Loading", International Journal of Fracture, Vol. 91, pp. 283-298, 1998.
10- Ayatollahi M.R., Pavier M.J., Smith D.J., "A New Specimen for Mode II Fracture tests", ECF14, Poland, Vol. 1, pp. 161-167, 2002.
11- Richard H.A., "A new shear specimen" International Journal of Fracture Vol. 17, pp. R105-R107, 1981.
12- Banks-Sills L., Arcan M. and Bui H.D. "Toward a pure shear specimen for K_{IIC} determination" International Journal of Fracture, Vol. 22, pp. R9-R13, 1983.