THE THIN PLATES ON ASYMMETRICAL THEORY OF ELASTICITY

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The modern technology makes it possible to study the microstructure of solid deformable bodies. The accumulated in this sphere facts certify the high role of the interior structure of the materials in processes, accompanying its deformation. A special interest in this field is connected with the development of new technological opportunities of not only researching and measuring the elements of the interior structure of the solid bodies, but also making an influence on this structure, and in the case of nano-technology- to create the demanded and necessary structural elements on microlevel. In this situation the development of the mathematical models gains actuality, as they can adequately describe the mechanical properties of such mediums and structures.
The research of the effects of microstructure of the elastic body on the basis of the structural phenomenological approach, i.e. on the basis of the mathematical continuumal models brought to the creation of the Asymmetrical Theory of Elasticity. The Asymmetrical (Momental, Micropolar) Theory of Elasticity, in present, is an essential extensive program of new development directions of the Theory of Deformation in Solid Bodies.
As in mathematical relation it's difficult to obtain the solution of the boundary value problem of the Three-dimensional Asymmetrical Theory of Elasticity and such solutions are in present obtained for a narrow range of problems, consequently, as in the Classical Theory of Elasticity, the problem of transition from the three-dimensional boundary value problems for thin plates and shells to the two-dimensional boundary value problems of the mathematical physics gains actuality.
In this work there develops the asymptotic method of the construction of the General Theories of Micropolar Elastic Thin Plates. The thin plates are considered both on the basis of the General Theory of Asymmetrical Elasticity with independent fields of rotation and transition, and on the basis of the Asymmetrical Theory of Elasticity with Straitened Rotation.
1. Let's consider the isotropic plate of constant 2h thickness as a three-dimensional elastic body and axis $(x_1\ x_2)$ of Cartesian system of coordinates $(x_1\ x_2\ x_3)$ and place them in the midplane of plate. We shall proceed from the basic equations of the three-dimensional problems of the Asymmetrical Theory of Elasticity with independent fields of rotation and transition [1].

Balance equations:

$$\sigma_{ji,j} = 0, \qquad \mu_{ji,j} + Э_{ijk}\sigma_{jk} = 0, \tag{1.1}$$

Physical correlation

$$\begin{cases} \sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}, \\ \mu_{ji} = (\gamma+\varepsilon)\chi_{ji} + (\gamma-\varepsilon)\gamma_{ij} + \beta\chi_{kk}\delta_{ij}; \end{cases} \tag{1.2}$$

Geometrical correlation

$$\gamma_{ij} = u_{j,i} - Э_{kij}\omega_k, \qquad \chi_{ij} = \omega_{j,i}, \tag{1.3}$$

where σ_{ij}, μ_{ij} are the components of force and momental stress tensors; γ_{ij}, χ_{ij} - the components of the asymmetrical deformation tensor and bending-torsion tensor; $\vec{u}$ -the vector of transition; $\vec{\omega}$ - the vector of independent turning of the body-plate; $\lambda, \mu, \alpha, \beta, \gamma, \varepsilon$ - the elastic constants of the material of plates; $Э_{ijk}$ - the components of pseudo tensor of Levi- Chivita, the comma in the indices means differentiation on the corresponding co- ordinate. Indexes i, j obtain 1, 2, 3 values.

On the plane of $x_3 = \pm h$ plates the force and momental boundary conditions are considered as given.

$$\sigma_{3i} = p_i^{\pm}, \ \mu_{3i} = m_i^{\pm} \quad (\mathrm{i, j} = 1,2,3) \ \text{when} \quad x_3 = \pm h. \tag{1.4}$$

In general case, the boundary conditions of mixed type are considered as given on the lateral cylindrical surface $(\Sigma = \Sigma_1 \cup \Sigma_2)$ with the vector of interior normal $\vec{n}$.

$$\begin{cases} \sigma_{ji}\cdot n_j = p_i^*, \mu_{ji}\cdot n_j = m_i^* \text{ on } \Sigma_1 \quad (i, j = 1, 2, 3), \\ \vec{u} = \vec{u}^*, \vec{\omega} = \vec{\omega}^* \text{ on } \Sigma_2 . \end{cases} \tag{1.5}$$

Let's also bring the solving system of the three-dimensional equations of Static Asymmetrical Theory of Elasticity with straitened rotation. This system is expressed as in works [2, 3].

Balance equations

$$\sigma_{ji,j} = 0, \qquad \mu_{ji,j} + Э_{ijk}\sigma_{jk} = 0; \tag{1.6}$$

Elasticity correlations

$$\begin{cases} \frac{1}{2}\left(\sigma_{ij}+\sigma_{ji}\right)=\frac{E}{1+\upsilon}\cdot e_{ij}+\frac{\upsilon\cdot E}{(1+\upsilon)(1-2\upsilon)}\cdot e_{kk}\cdot\delta_{ij}\ , \\ \mu_{ij}=B_1\cdot k_{ij}+B_2\cdot k_{ji}\ ; \end{cases} \tag{1.7}$$

Geometrical correlations

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right),\qquad k_{ij}=\omega_{j,i}\ ,\quad \omega_i=\frac{1}{2}Э_{ijk}u_{k,j}\ . \tag{1.8}$$

The solution of the set boundary value problems for plates (1.1)-(1.5), (1.6)-(1.8) is formed from the sum of solutions of symmetrical of x_3 (generalized plane stress state) and asymmetrical (bending) problems. In the symmetrical problem the even values by x_3 will be $\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{21},\mu_{13},\mu_{23},\mu_{31},\mu_{32},u_1,u_2,\omega_3$ and the odd ones are $\sigma_{31},\sigma_{13},\sigma_{23},\sigma_{32},\mu_{11},\mu_{22},\mu_{33},\mu_{12},\mu_{21},\ \omega_1,\omega_2,u_3$ and in the case of asymmetrical problem it'll be vice-verse.

It's assumed that the thickness of plate is small in reference to its characteristic size a in the midplane ($2h << a,\ \delta=h/a << 1$- small geometrical parameter). The property of the three-dimensional plate expressed by formula [4] lies in the basis of the judgments:

$$(\text{SDS})_{\text{complete}}=(\text{SDS})_{\text{interior}}+(\text{SDS})_{\text{boundary}}\ . \tag{1.9}$$

Here, by $(\text{SDS})_{\text{complete}}$, $(\text{SDS})_{\text{interior}}$ and $(\text{SDS})_{\text{boundary}}$ we mean the complete, interior (penetrating) and boundary stress-deformed states. The principle problem of consideration is that of the approximate methods of the interior accounts of plates and also of the boundary accounts. Here, both of these methods are constructed on the basis of the asymptotic integration of the three- dimensional linear differential equations of the Static Theory of the Asymmetrical Theory of Elasticity with independent fields of rotation, transition and also with straitened rotation.

3. Let's consider the General Theory of Micropolar Thin Plates on the basis of the Asymmetrical Theory of Elasticity with independent fields of rotation, transition.

For the construction of $(\text{SDS})_{\text{interior}}$ in equations (1.1)-(1.4) let's replace the independent variables by the following formula:

$$\xi=\frac{x_1}{a}\ ,\qquad \eta=\frac{x_2}{a}\ ,\qquad \zeta=\frac{x_3}{h}\ . \tag{2.1}$$

and let's find the solution of the interior problem in the following view:

$$Q^{вн}=\delta^{-q}\sum\delta^{s}Q^{(s)}\ . \tag{2.2}$$

Here, Q – is any of the stress (momental, force), rotation and transition, value q- a natural number, which varies for different values and which is defined by the condition of obtaining the non- contradicting recurrent systems of equations in the asymptotical approximations. Thus, we'll have the following:

for the asymmetrical by x_3 problem (the bending of plates on the Asymmetrical Theory of Elasticity):

$$\begin{cases} q=1\ \text{ for }\ \sigma_{i3}\ ,\sigma_{3i}\ (i=1,2),\ \mu_{mn}\ (mn:11,22,33,12,21),\ \omega_i\ (i=1,2),\ u_3\ , \\ q=0\ \text{ for }\sigma_{mn}\ (mn:11,22,33,12,21),\ \mu_{i3}\ ,\ \mu_{3i}\ (i=1,2),\ u_i\ (i=1,2),\ \omega_3\ ; \end{cases} \tag{2.3}$$

for the symmetrical by x_3 (generalized plane stress state of plate on the Asymmetrical Theory of Elasticity):

$$\begin{cases} q=2\ \text{ for}\sigma_{mn}\ \ (mn:11,22,12,21),\ \mu_{i3}\ ,\mu_{3i}\ \ (i=1,2),\ u_i\ (i=1,2),\ \omega_3\ , \\ q=1\ \text{ for }\sigma_{i3}\ ,\sigma_{3i}\ (i=1,2),\ \mu_{mn}\ (mn:11,22,33,12,21),\ \omega_i\ (i=1,2),\ u_3\ , \\ q=0\ \text{ for }\sigma_{33}. \end{cases} \tag{2.4}$$

Now, let's bring the results of the initial asymptotical approximations of interior problem, i.e. the equations of the General Applied Two-dimensional Theory of Micropolar Thin Plates on the basis of the Asymmetrical Theory of Elasticity with independent fields of transition and rotation:

Balance equations

$$\begin{cases} \frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}=-\left(p_3^{+}+p_3^{-}\right), \\ \frac{\partial L_{11}}{\partial x_1}+\frac{\partial L_{21}}{\partial x_2}+N_{23}=-\left(m_1^{+}+m_1^{-}\right); \end{cases} \tag{2.5}$$

Elasticity correlations

$$\begin{cases} N_{13} = 2h\left[(\mu+\alpha)\Gamma_{13} + (\mu-\alpha)\Gamma_{31}\right], N_{31} = 2h\left[(\mu+\alpha)\Gamma_{31} + (\mu-\alpha)\Gamma_{13}\right] \ (1 \to 2), \\ L_{11} = 2h\left[(2\gamma+\beta)k_{11} + \beta\left(k_{22}+k_{33}\right)\right] \ (11 \to 22 \to 33), \quad L_{33} = h\left(m_3^+ - m_3^-\right), \\ L_{12} = 2h\left[(\gamma+\varepsilon)k_{12} + (\gamma-\varepsilon)k_{21}\right] \ (1 \rightleftarrows 2), \ N_{32} = h\left(p_2^+ - p_2^-\right) \ (1 \rightleftarrows 2); \end{cases} \tag{2.6}$$

Geometrical correlations

$$\begin{cases} \Gamma_{13} = \dfrac{\partial w}{\partial x_1} + O_2 \,, \ \Gamma_{23} = \dfrac{\partial w}{\partial x_2} - O_1 \,, \\ k_{11} = \dfrac{\partial O_1}{\partial x_1} (1 \to 2) \,, k_{12} = \dfrac{\partial O_2}{\partial x_1} \ (1 \rightleftarrows 2). \end{cases} \tag{2.7}$$

Here, N_{i3} $(i = 1,2)$, L_{mn} $(mn : 11,22,12,21)$ are the averaged by thickness plates of force and moments:

$$N_{i3} = \int_{-h}^{h} \sigma_{i3} dx_3 \ , \quad L_{mn} = \int_{-h}^{h} \mu_{mn} dx_3 \tag{2.8}$$

Γ_{i3} $(i = 1,2)$ are the components of deformation tensor, k_{mn} $(mn : 11,22,12,21)$ - the components of bending- torsion in the points of the midplane of plate, w-deflection of points of the midplane of plate; O_1 и O_2 - the turnings of the mentioned points around the axis x_1 и x_2.

In this work there is investigated the problem of constructing the boundary layer with independent fields of transition and rotation. The properties and the structure of the boundary layer of the micropolar plate. The existence of four types of boundary layers (force-plane and antiplane, momental-plane and antiplane) is proved. There are constructed functions of boundary layer type.

The problem of matching of the interior problem (of the Applied Theory) and the boundary layers with the aim of satisfaction to the three-dimensional boundary conditions of the Asymmetrical Theory of Elasticity on the lateral cylindrical surface of plate is also researched. In the result of this research there'll be formulated the boundary conditions of the Applied Theory. Separate boundary conditions are obtained also for the structural problems of boundary layers of micropolar thin plates.

Let's bring the boundary conditions of the General Applied Theory of Bending (2.5)-(2.7) of micropolar thin plates (for the first boundary value problem of the Asymmetrical Theory of Elasticity):

$$N_{13}\Big|_{x_1 = const} = N_{13}^* \ , \quad L_{11}\Big|_{x_1 = const} = L_{11}^* \ , \quad L_{12}\Big|_{x_1 = const} = L_{12}^* \ . \tag{2.9}$$

The system of equations (2.5)-(2.7) with boundary conditions (2.9) constitute the General Theory of Micropolar Thin Plates (on the basis of the Asymmetrical Theory of Elasticity with independent fields of transition and rotation):

Balance equations

$$\begin{cases} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial S_{21}}{\partial x_2} = -\left(p_1^+ + p_1^-\right) \ (1 \rightleftarrows 2), \\ \dfrac{\partial L_{13}}{\partial x_1} + \dfrac{\partial L_{23}}{\partial x_2} + S_{12} - S_{21} = -\left(m_3^+ + m_3^-\right); \end{cases} \tag{2.10}$$

Elasticity correlations

$$\begin{cases} T_{11} = \dfrac{2Eh}{1-\upsilon^2}\left[\Gamma_{11} + \upsilon\Gamma_{22}\right] \,, \ \overset{(s)}{S_{12}} = 2h\left[(\mu+\alpha)\Gamma_{12} + (\mu-\alpha)\Gamma_{21}\right] \ (1 \leftrightarrows 2), \\ L_{13} = 2h\left[(\gamma+\varepsilon)k_{13} + (\gamma-\varepsilon)k_{31}\right] \,, \ L_{31} = h\left(m_1^+ - m_1^-\right) = 2h\left[(\gamma+\varepsilon)k_{31} + (\gamma-\varepsilon)k_{13}\right] \ (1 \leftrightarrows 2); \end{cases} \tag{2.11}$$

Geometrical correlations

$$\Gamma_{11} = \frac{\partial V_1}{\partial x_1}(1 \to 2), \ \Gamma_{12} = \frac{\partial V_2}{\partial x_1} - O_3, \ \Gamma_{21} = \frac{\partial V_1}{\partial x_2} + O_3, \ k_{13} = \frac{\partial O_3}{\partial x_1} \ (1 \to 2). \tag{2.12}$$

Here T_{11} $(1 \to 2)$, S_{12} $(1 \rightleftarrows 2)$, L_{13} $(1 \to 2)$ are the averaged by thickness plates of force and moments.

$$T_{11} = \int_{-h}^{h} \sigma_{11} dx_3 \ , \ S_{12} = \int_{-h}^{h} \sigma_{12} dx_3 \ , \ L_{13} = \int_{-h}^{h} \mu_{13} dx_3 \tag{2.13}$$

Γ_{11} $(1 \to 2)$, Γ_{12} $(1 \rightleftarrows 2)$ are the components of deformation tensor; k_{13} $(1 \to 2)$ components of bending- torsion points of the midplane of plate; V_i $(i = 1,2)$ -the transitions; O_3 -turning around the axis x_3 of points of the midplane of plate.
On the basis of matching of the asymptotical expansion of the interior problem and boundary layers we'll obtain the boundary conditions of the interior problem:

$$T_{11} \Big|_{x_1 = const} = T_{11}^* \ , \quad S_{12} \Big|_{x_1 = const} = S_{12}^* \ , \quad L_{13} \Big|_{x_1 = const} = L_{13}^* \ . \tag{2.14}$$

The General Theory of the Generalized Plane Stress State of Micropolar Thin Elastic Plates is defined by the (2.10)-(2.12) system of equations and by the boundary conditions (2.14).

3. Let's consider the General Theory of Micropolal Thin Plates on the basis of the Asymmetrical Theory of Elasticity with straitened rotation.
By choosing the corresponding q value in formula (2.2) there is built the asymptotica of the interior problem of the Asymmetrical Theory of Elasticity with straitened rotation (1.6) - (1.8).
On the basis of the initial approximation of the asymptotical method for the interior problem we can bring the solving equations of the Applied Two-dimensional Theory of Micropolar Plates with straitened rotation.
The bending problem:
In this case the solving equations are generalized Sophie- Germain- Lagrange equations on micropolar case:

$$\left(\overset{cl.}{D} + \overset{cup.}{D} \right) \nabla^2 \nabla^2 w = -\left(p_3^+ + p_3^-\right), \tag{3.1}$$

where, $\overset{cl.}{D} = \frac{2Eh^3}{3(1-\nu^2)}$ is the classical rigidity of the plate, and $\overset{cup.}{D} = 2h(\gamma + \varepsilon)$ the momental rigidity of the plate. Let's mention that on the basis of special asymptotica (close to the classical asymptotica) it's possible to obtain equation (3.1) proceeding from the General Theory of the Asymmetrical Theory of Elasticity with independent fields of rotation and transition (1.1) –(1.3).
There is also constructed the boundary layer and there is also considered the problem of matching of the asymptotical expansion of the interior problem and of the boundary layer in the Theory of Asymmetrical Theory of Elasticity with straitened rotation and, finally, there are formulated the boundary conditions for (3.1)equation:

$$\left(M_{11}^{(0)} + L_{12}^{(0)}\right) \Big|_{x_1 = const} = \int_{-h}^{h} \left(x_3 p_1^* + m_2^*\right) dx_3 \ , \quad N_{13}^{(0)} + \frac{\partial}{\partial x_2}\left(M_{12}^{(0)} - L_{11}^{(0)}\right) \Big|_{x_1 = const} = \int_{-h}^{h} \left[p_3^* + \frac{\partial}{\partial x_2}\left(x_3 p_2^* - m_1^*\right) \right] dx_3 \ . \tag{3.2}$$

where,

$$\begin{cases} M_{11} = -\frac{2}{3}\frac{Eh^3}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x_1^2} + \upsilon \frac{\partial^2 w}{\partial x_2^2}\right), \ M_{12} = -\frac{2}{3}\frac{Eh^3}{1+\upsilon}\frac{\partial^2 w}{\partial x_1 \partial x_2}, \\ L_{11} = 4\gamma \frac{\partial^2 w}{\partial x_1 \partial x_2}, \ L_{12} = 2h\left[(\gamma - \varepsilon)\frac{\partial^2 w}{\partial x_2^2} - (\gamma + \varepsilon)\frac{\partial^2 w}{\partial x_1^2}\right], \\ N_{13} = -\left[2h(\gamma + \varepsilon) + \frac{2}{3}\frac{Eh^3}{1-\upsilon^2}\right]\frac{\partial}{\partial x_1}\left(\nabla^2 w\right) + 2\left(hX_2 + m_2\right). \end{cases} \tag{3.3}$$

The generalized plane stress state:
The solving system of equations of the General Applied Theory of Micropolar Thin Plate with straitened rotation on the basis of the initial approximation of the interior process takes the following shape:

$$
\begin{cases}
2\dfrac{\partial^2 V_1}{\partial x_1^2}+\dfrac{2\upsilon}{1-\upsilon}\left(\dfrac{\partial^2 V_1}{\partial x_1^2}+\dfrac{\partial^2 V_2}{\partial x_1 \partial x_2}\right)+\left(\dfrac{\partial^2 V_1}{\partial x_2^2}+\dfrac{\partial^2 V_2}{\partial x_1 \partial x_2}\right)+\dfrac{1}{\mu}\dfrac{\gamma\varepsilon}{\gamma+\varepsilon}\nabla^2\left(\dfrac{\partial^2 V_2}{\partial x_1 \partial x_2}-\dfrac{\partial^2 V_1}{\partial x_2^2}\right)= \\
=-\dfrac{1}{\mu}\left\langle \dfrac{1}{h}X_1+\dfrac{1}{2}\dfrac{\partial}{\partial x_2}\left[\dfrac{1}{h}q_1+\dfrac{\gamma-\varepsilon}{\gamma+\varepsilon}\left(\dfrac{\partial m_1}{\partial x_1}+\dfrac{\partial p_1}{\partial x_2}\right)\right]\right\rangle, \\
2\dfrac{\partial^2 V_2}{\partial x_2^2}+\dfrac{2\upsilon}{1-\upsilon}\left(\dfrac{\partial^2 V_2}{\partial x_2^2}+\dfrac{\partial^2 V_1}{\partial x_1 \partial x_2}\right)+\left(\dfrac{\partial^2 V_2}{\partial x_1^2}+\dfrac{\partial^2 V_1}{\partial x_1 \partial x_2}\right)-\dfrac{1}{\mu}\dfrac{\gamma\varepsilon}{\gamma+\varepsilon}\nabla^2\left(\dfrac{\partial^2 V_2}{\partial x_1^2}-\dfrac{\partial^2 V_1}{\partial x_1 \partial x_2}\right)= \\
=-\dfrac{1}{\mu}\left\langle \dfrac{1}{h}Y_1-\dfrac{1}{2}\dfrac{\partial}{\partial x_1}\left[\dfrac{1}{h}q_1+\dfrac{\gamma-\varepsilon}{\gamma+\varepsilon}\left(\dfrac{\partial m_1}{\partial x_1}+\dfrac{\partial p_1}{\partial x_2}\right)\right]\right\rangle.
\end{cases}
\qquad (3.4)
$$

To this system it's also necessary to attach the corresponding boundary conditions which are obtained by matching the asymptotical expansion of the interior problem with the problem of the boundary layer.

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