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THE TUNNEL-CRACK WITH A SLIGHTLY WAVY FRONT UNDER SHEAR LOADING

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ABSTRACT

One considers a planar tunnel-crack with a slightly wavy front in an infinite body, loaded in mode II+III through uniform remote shear stresses, such as a geophysical fault. The distribution of stress intensity factors along the perturbed front is determined using Bueckner-Rice's weight function theory. From there, one addresses the following bifurcation and stability problems: (i) is there a non-rectilinear configuration of the front for which the energy release rate is uniform along that front? (ii) is the rectilinear configuration of the front stable versus small coplanar perturbations? The answer to question (i) is positive. The "critical", bifurcated configuration is sinusoidal; both its wavelength and the "phase difference" between the fore and rear parts of the crack front depend upon the ratio of the initial (prior to perturbation of the front) mode II and III stress intensity factors. The answer to question (ii) depends upon the wavelength of the perturbation envisaged; stability prevails for wavelengths smaller than the critical one and instability for larger ones. This conclusion is similar to those arrived at by Gao and Rice and the authors for analogous problems.

KEYWORDS

Tunnel-crack, slightly perturbed front, shear loading, Bueckner-Rice theory, bifurcation, stability

INTRODUCTION

Consider a plane crack with arbitrary contour $\mathcal{F}$ in an arbitrary body Ω. Slightly perturb the crack front, within the crack plane, by an amount $\delta a(s)$, where s denotes the curvilinear distance along $\mathcal{F}$. Then the variations $\delta K_\alpha(s)$ ($\alpha = I, II, III$) of the stress intensity factors (SIF) are given, to first order in the perturbation, by the following formula, which was first established by Nazarov [1] and Rice [2] in situations of pure mode I, and later extended by Leblond *et al.* [3] to arbitrary mixed mode conditions:

$$\begin{aligned}\delta K_\alpha(s) = [\delta K_\alpha(s)]_{\delta a(s') \equiv \delta a(s)} + N_{\alpha\beta}(0) K_\beta(s) \frac{\mathrm{d}\delta a}{\mathrm{d}s}(s) \\ + PV \int_{\mathcal{F}} Z_{\alpha\beta}(\Omega; s, s') K_\beta(s') [\delta a(s') - \delta a(s)] ds'\end{aligned} \tag{1}$$

where Einstein's implicit summation convention is employed for the index $\beta = I, II, III$. In this equation, the $K_\beta(s)$ are the initial (prior to perturbation of the crack front) SIF; $[\delta K_\alpha(s)]_{\delta a(s') \equiv \delta a(s)}$ denotes the value of $\delta K_\alpha(s)$ for a *uniform crack advance* equal to $\delta a(s)$ ($\delta a(s') = \delta a(s), \forall s'$); the $N_{\alpha\beta}(0)$ are the components of a *universal* (valid in all circumstances) operator which has been calculated by Gao and Rice [4]; and finally the $Z_{\alpha\beta}(\Omega; s, s')$ are the components of an operator which depends upon (in addition to s and s') the entire geometry of the body and the crack considered, and diverges like $(s' - s)^{-2}$ for $s' \to s$, so that the integral in Eqn. 1 makes sense as a Cauchy principal value (PV).

Equation 1 was applied by Leblond *et al.* [5] to the study of a planar tunnel-crack with a slightly wavy front in an infinite body, loaded in pure mode I through some uniform remote tensile stress. Using an original method based on the work of Rice [2], these authors first evaluated the geometry-dependent operator component $Z_{I,I}(\Omega; s, s')$ for the configuration envisaged. Then they studied problems of configurational bifurcation and stability of the crack front during propagation. The bifurcation problem is the following one. Does there exist, in addition to the trivial, rectilinear configuration of both parts of the crack front, some non-trivial, curved configuration for which the energy release rate is uniform along that front in spite of its curvature? The answer was shown to be "yes"; the "critical", bifurcated configuration was symmetric with respect to the middle axis of the tunnel-crack and sinusoidal, its wavelength being a characteristic multiple of the crack width. The stability issue was as follows: if both parts of the crack front are slightly perturbed within the crack plane, will the perturbation decay or increase as propagation proceeds? It was shown that stability prevails for sinusoidal perturbations of wavelength smaller than that of the critical perturbation, and instability for wavelengths larger than it. This finding was compatible with the conclusions of Rice [6] and Gao and Rice [7, 8] concerning other types of cracks loaded in mode I.

The aim of this paper is to consider the same problem, but for a shear (mode II+III) loading. Propagation will still be assumed to be coplanar; this is reasonable provided that the crack is channeled along a planar surface of low fracture resistance, which can be the case for instance for a geological fault. Also, propagation will be considered to be governed by the (local) energy release rate, the critical value of which will be assumed to be independent of the ratio of the mode II and III SIF. Again, this is reasonable (Rice, private communication) for coplanar propagation along a weak surface, since energy dissipation occurs through the same physical mechanisms (shear and friction) in both modes II and III. It will be shown that there again exists a critical, bifurcated configuration of the front. Again, this configuration is sinusoidal and its wavelength is a multiple of the width of the crack, but this wavelength now depends upon the ratio of the mode II and III initial (prior to perturbation of the front) SIF. Also, it is symmetric with respect to the middle axis of the crack only for initial conditions of pure mode II or pure mode III; for mixed mode II+III conditions, there is a "phase difference" between the bifurcated configurations of the fore and rear parts of the crack front. The stability issue will be addressed only in the case where the phase difference between the perturbations of both parts of the front takes some special values. It will be shown that in the most interesting case, stability prevails only if the wavelength of the perturbation is smaller than the critical one. This conclusion is the same as in pure mode I (Leblond *et al.* [5]), and also as in mixed mode for other crack shapes (Gao and Rice [9], Gao [10]).

STRESS INTENSITY FACTORS FOR A PERTURBED TUNNEL-CRACK

Consider now (Figure 1), within an infinite body, a tunnel-crack of half-width a loaded through uniform remote shear stresses $\sigma_{xy}^{\infty}, \sigma_{yz}^{\infty}$. The orientations of the fore (+) and rear (−) parts of the crack front being chosen as identical, this loading generates a uniform SIF $K_{II} = \sigma_{xy}^{\infty}\sqrt{\pi a}$ and opposite SIF $K_{III}^{+} = \sigma_{yz}^{\infty}\sqrt{\pi a}$, $K_{III}^{-} = -K_{III}^{+}$ on them. Now slightly perturb the fore and rear parts of the crack front, within the crack plane, by the amounts $\delta a(z^{+})$, $\delta a(z^{-})$ respectively. Using then Eqn. 1, the values of the $N_{\alpha\beta}(0)$ provided by Gao and Rice [4], "symmetry" properties of the $Z_{\alpha\beta}(\Omega; s, s')$ established by Leblond *et al.* [3] and elementary symmetry considerations for the crack configuration envisaged, one gets for the perturbations of the SIF on the fore part of the crack front:

$$\begin{aligned}
\delta K_{II}(z^{+}) = K_{II}\frac{\delta a(z^{+})}{4a} - \frac{2}{2-\nu}K_{III}^{+}\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^{+}) \\
+PV\int_{-\infty}^{+\infty}\left[f_{II,II}\left(\frac{z'-z}{a}\right)K_{II} + f_{II,III}\left(\frac{z'-z}{a}\right)K_{III}^{+}\right]\left(\delta a(z'^{+}) - \delta a(z^{+})\right)\frac{dz'}{(z'-z)^2} \\
+\int_{-\infty}^{+\infty}\left[g_{II,II}\left(\frac{z'-z}{a}\right)K_{II} + g_{II,III}\left(\frac{z'-z}{a}\right)K_{III}^{-}\right]\delta a(z'^{-})\frac{dz'}{a^2}\ ;
\end{aligned} \tag{2}$$

$$\begin{aligned}
\delta K_{III}(z^{+}) = K_{III}^{+}\frac{\delta a(z^{+})}{4a} + \frac{2(1-\nu)}{2-\nu}K_{II}\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^{+}) \\
+PV\int_{-\infty}^{+\infty}\left[-(1-\nu)f_{II,III}\left(\frac{z'-z}{a}\right)K_{II} + f_{III,III}\left(\frac{z'-z}{a}\right)K_{III}^{+}\right]\left(\delta a(z'^{+}) - \delta a(z^{+})\right)\times \\
\times\frac{dz'}{(z'-z)^2} + \int_{-\infty}^{+\infty}\left[-(1-\nu)g_{II,III}\left(\frac{z'-z}{a}\right)K_{II} + g_{III,III}\left(\frac{z'-z}{a}\right)K_{III}^{-}\right]\delta a(z'^{-})\frac{dz'}{a^2}\ .
\end{aligned} \tag{3}$$

In these expressions, the $f_{\alpha\beta}$ and $g_{\alpha\beta}$ are functions which depend solely upon (in addition to the argument $(z'-z)/a$) Poisson's ratio ν; $f_{II,II}$, $f_{III,III}$, $g_{II,II}$, $g_{III,III}$ are even, and $f_{II,III}$, $g_{II,III3}$ odd. The values of $\delta K_{II}(z^-)$, $\delta K_{III}(z^-)$ are given by the same expressions, with the obvious substitutions $\delta a(z^+) \to \delta a(z^-)$, $\delta a(z'^{\pm}) \to \delta a(z'^{\mp})$, $K^{\pm}_{III} \to K^{\mp}_{III}$. The functions $f_{\alpha\beta}$ and $g_{\alpha\beta}$ ($\alpha, \beta = II, III$) can be

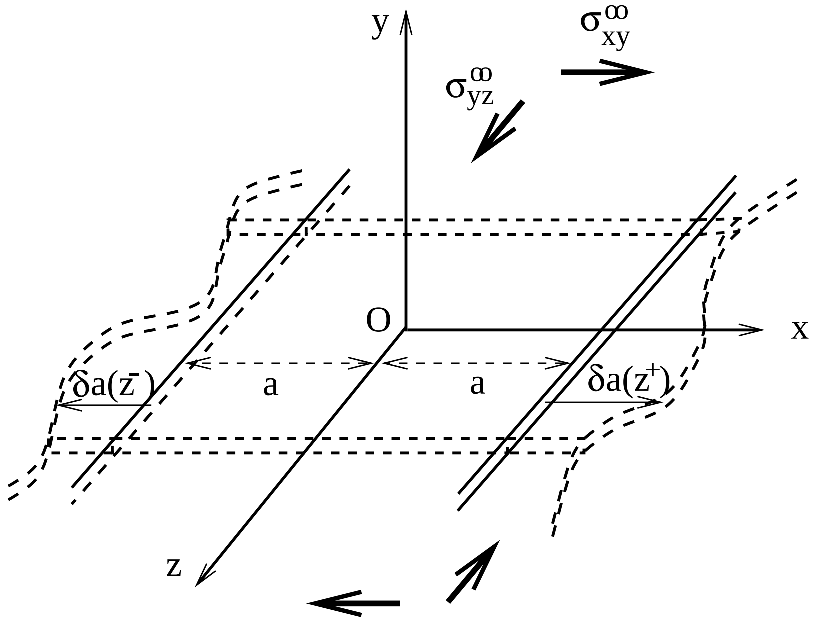

Figure 1: Tunnel-crack with a slightly wavy front loaded in shear

determined in a similar way as the corresponding functions for mode I (Leblond *et al.* [5]). The method combines the relation between the $Z_{\alpha\beta}(\Omega; s, s')$ and Bueckner's crack-face weight functions (Leblond *et al.* [3]), and Eqn. 1 applied to some special motions of the crack front preserving the shape of the crack while modifying its size and orientation. It yields integro-differential equations on the $f_{\alpha\beta}$ and $g_{\alpha\beta}$, which are transformed into ordinary differential equations through Fourier transform along the $z-$direction, and then integrated numerically for any value of ν.

BIFURCATION OF THE STRAIGHT CONFIGURATION OF THE FRONT

Consider a slightly curved configuration of both parts of the crack front defined by

$$\delta a(z^+) = \alpha \cos(kz) \; ; \; \delta a(z^-) = \alpha \cos(kz + \varphi) \tag{4}$$

where α, $k > 0$ and $\varphi \in [-\pi, \pi)$ are parameters. Using Eqns. 2 and 3 and Irwin's formula, one obtains the following expression of the perturbation of the energy release rate $\mathcal{G}$ on the fore part of the crack front:

$$\delta\mathcal{G}(z^+) = 2\frac{1-\nu^2}{E}\frac{\alpha}{a}K^2_{II}[(F + G\cos\varphi + H\sin\varphi)\cos(kz) + (-G\sin\varphi + H\cos\varphi)\sin(kz)] \; . \tag{5}$$

In this expression, E is Young's modulus and $F \equiv F(K^+_{III}/K_{II}, p)$, $G \equiv G(K^+_{III}/K_{II}, p)$, $H \equiv H(K^+_{III}/K_{II}, p)$ the quantities given by

$$\begin{gathered} F = \bar{f}_{II,II}(p) + \frac{1}{1-\nu}\frac{K^{+2}_{III}}{K^2_{II}}\bar{f}_{III,III}(p) \; ; \\ G = \bar{g}_{II,II}(p) - \frac{1}{1-\nu}\frac{K^{+2}_{III}}{K^2_{II}}\bar{g}_{III,III}(p) \; ; \; H = 2\frac{K^+_{III}}{K_{II}}\bar{g}_{II,III}(p) \end{gathered} \tag{6}$$

where the "reduced" wavevector p (> 0) and the functions $\bar{f}_{\alpha\beta}$, $\bar{g}_{\alpha\beta}$ are defined by

$$\begin{gathered} p = ka \; ; \; \bar{f}_{\alpha\beta}(p) = \frac{1}{4} + 2\int_0^{+\infty} f_{\alpha\beta}(u)(\cos(pu) - 1)\frac{du}{u^2} \; , \\ \bar{g}_{\alpha\beta}(p) = 2\int_0^{+\infty} g_{\alpha\beta}(u)\cos(pu)du \qquad ((\alpha,\beta) = (II,II) \text{ and } (III,III)) \; ; \\ \bar{g}_{II,III}(p) = 2\int_0^{+\infty} g_{II,III}(u)\sin(pu)du \; . \end{gathered} \tag{7}$$

The expression of $\delta\mathcal{G}(z^-)$ is given by the same formula 5 as $\delta\mathcal{G}(z^+)$ with the substitutions $\cos(kz) \to \cos(kz+\varphi)$, $\sin(kz) \to -\sin(kz+\varphi)$.

For $\mathcal{G}$ to be uniform along both parts of the crack front, the terms proportional to $\cos(kz)$ and $\sin(kz)$ in the expression of $\delta\mathcal{G}(z^+)$, and those proportional to $\cos(kz+\varphi)$ and $\sin(kz+\varphi)$ in the expression of $\delta\mathcal{G}(z^-)$, must be zero. This leads to the following conditions:

$$F + G\cos\varphi + H\sin\varphi = 0\ ;\ \tan\varphi = H/G\ . \tag{8}$$

Using Eqn. 8_2 in Eqn. 8_1, one gets $\cos\varphi = -FG/(G^2+H^2)$, $\sin\varphi = -FH/(G^2+H^2)$. Use of the relation $\cos^2\varphi + \sin^2\varphi = 1$ then yields

$$F^2 = G^2 + H^2 \quad\Rightarrow\quad F = \pm\sqrt{G^2+H^2}\ ;\ \cos\varphi = -\frac{G}{F}\ ;\ \sin\varphi = -\frac{H}{F}\ . \tag{9}$$

For a given ratio K^+_{III}/K_{II}, 9_2 is an equation on p the solution of which is the critical reduced wavevector; Eqns. 9_3, 9_4 then define the corresponding critical phase difference between the configurations of the fore and rear parts of the crack front.

It can be shown that for $p=0$, $F = G = \frac{1}{4}\left(1 + \frac{1}{1-\nu}\frac{K^{+2}_{III}}{K^2_{II}}\right)$, $H=0$, and that for $p \to +\infty$, $F \to -\infty$, $G \to 0$, $H \to 0$. Therefore, if one chooses the sign $+$ in Eqn. 9_2, the solution is obviously $p=0$, and it then follows from Eqns. 9_3, 9_4 that $\varphi = -\pi$, so that by Eqns. 4, $\delta a(z^+) = -\delta a(z^-) = Cst$. This is a trivial bifurcation mode which merely corresponds to some translatory motion of the crack in the $x-$direction. On the other hand, if the sign $-$ is selected in Eqn. 9_2, there is a non-zero solution p_c and a corresponding angle φ_c, which define a non-trivial bifurcation mode. It can be shown that $\cos\varphi_c = -G/F > 0$ so that $\varphi_c \in (-\pi/2, \pi/2)$.

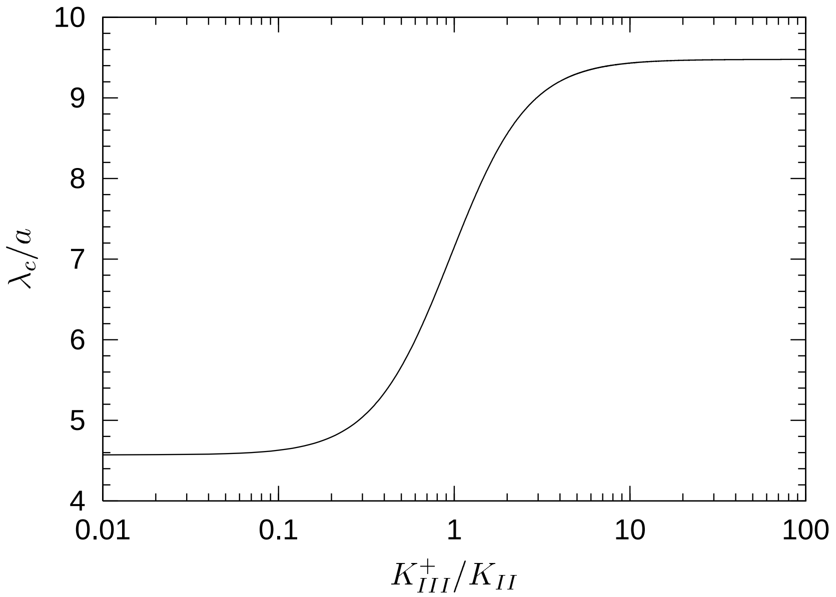

Figure 2: Critical reduced wavelength versus the ratio of the initial SIF

For each value of the ratio K^+_{III}/K_{II}, Eqn. 9_2 (with the sign $-$) can be solved numerically, using the values of the functions $f_{\alpha\beta}$, $g_{\alpha\beta}$ determined as sketched in the preceding section. Figures 2 and 3 represent the critical reduced wavelength $\lambda_c/a = 2\pi/p_c$ and the critical phase difference φ_c of the bifurcated mode, as functions of this ratio, for $\nu = 0.3$. (K^+_{III}/K_{II} is assumed here to be positive; it is obvious that if it changes sign, λ_c remains unchanged while φ_c changes sign). One sees that the critical wavelength is larger in pure mode III than in pure mode II. Also, the critical phase difference vanishes in pure mode II and III, that is, the bifurcated configuration becomes symmetric with respect to the middle axis Oz of the crack in these cases. It is recalled that the bifurcation mode was also found to be symmetric for a pure mode I loading (Leblond *et al.* [5]).

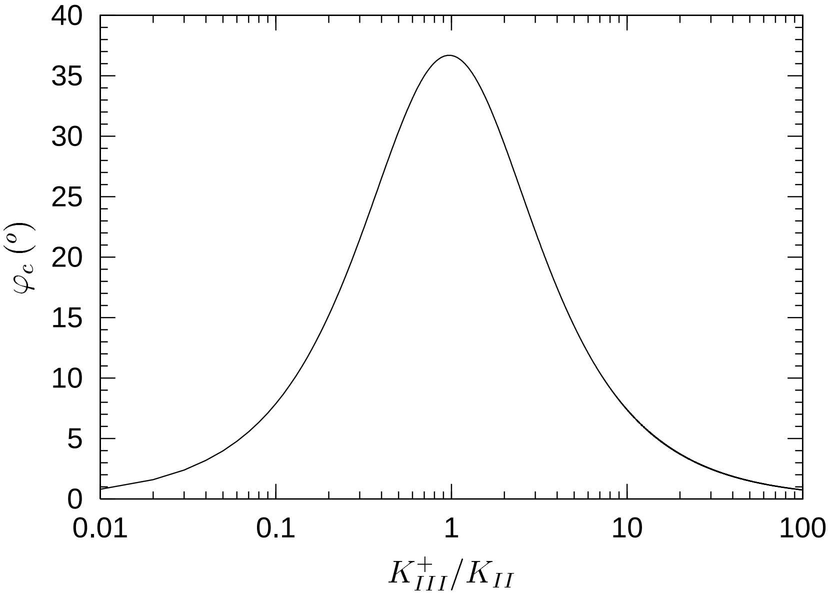

Figure 3: Critical phase difference versus the ratio of the initial SIF

STABILITY OF THE STRAIGHT CONFIGURATION OF THE FRONT

The issue of configurational stability of the crack front is easily dealt with only if the extrema of $\delta\mathcal{G}(z^+)$ coincide with those of $\delta a(z^+)$, and similarly for those of $\delta\mathcal{G}(z^-)$ and $\delta a(z^-)$; then one gets stability if the maxima of $\delta\mathcal{G}(z^+)$ and $\delta\mathcal{G}(z^-)$ correspond to the minima of $\delta a(z^+)$ and $\delta a(z^-)$, and instability if they correspond to the maxima of $\delta a(z^+)$ and $\delta a(z^-)$. We thus assume that the terms proportional to $\sin(kz)$ and $\sin(kz+\varphi)$ in the expressions of $\delta\mathcal{G}(z^+)$ and $\delta\mathcal{G}(z^-)$ vanish, *i.e.* that the phase difference φ is given (in terms of K^+_{III}/K_{II} and p) by Eqn. 8_2. Then stability prevails if the cofactors of $\cos(kz)$ and $\cos(kz+\varphi)$ in the expressions of $\delta\mathcal{G}(z^+)$ and $\delta\mathcal{G}(z^-)$ are negative:

$$\text{Stability} \quad \Leftrightarrow \quad F + G\cos\,\varphi + H\sin\,\varphi < 0 \;\; (\text{with } \tan\,\varphi = H/G)\;. \tag{10}$$

Let us for instance assume K^+_{III}/K_{II} to be positive. Then it can be checked that $\tan\,\varphi = H/G \geq 0$ so that $\varphi \in [0,\pi/2)$ or $\varphi \in [-\pi,-\pi/2)$. Note that if $p = p_c$, $\varphi = \varphi_c$ in the first case and $\varphi = \varphi_c - \pi$ in the second one.

* The more interesting case corresponds to $\varphi \in [0,\pi/2)$. Then, for $p = 0$, $F = G > 0$ and $H = 0$ (see above) so that $\varphi = 0$ and $F + G\cos\,\varphi + H\sin\,\varphi = F + G > 0$. On the other hand, for $p \to +\infty$, $F \to -\infty$, $G \to 0$, $H \to 0$ (see above) so that $F + G\cos\,\varphi + H\sin\,\varphi \sim F < 0$. Finally, for $p = p_c$, $\varphi = \varphi_c$ so that $F + G\cos\,\varphi + H\sin\,\varphi = 0$. Thus $F + G\cos\,\varphi + H\sin\,\varphi$ is positive for $p < p_c$, zero for $p = p_c$ and negative for $p > p_c$: *stability prevails for wavelengths smaller than the critical value λ_c and instability for wavelengths greater than it.* This finding is similar to those of Leblond *et al.* [5] in pure mode I, and Gao and Rice [9] and Gao [10] for semi-infinite and penny-shaped cracks in mode II+III.

* In the less interesting case where $\varphi \in [-\pi,-\pi/2)$, for $p = 0$, $\varphi = -\pi$ so that $F + G\cos\,\varphi + H\sin\,\varphi = F - G = 0$; for $p \to +\infty$, $F + G\cos\,\varphi + H\sin\,\varphi < 0$; finally, for $p = p_c$, $\varphi = \varphi_c - \pi$ so that $F + G\cos\,\varphi + H\sin\,\varphi = F + G^2/F + H^2/F = -2\sqrt{G^2+H^2} < 0$. Thus $F + G\cos\,\varphi + H\sin\,\varphi$ is always negative, and stability prevails for all wavelengths.

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